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MATH 301 | Differential equations | 2024

Assignment 1: Partial Differential Equations — PDEs

Set: Monday 4 March 2024 Due: Friday 15 March 2024 at 23:59 (End of week 3)

1. **Classification questions (easy):**

   Determine the order of the following PDEs for a function $U(x,y)$, $u(x,y)$, $\psi(x,t)$, or $\Psi(x,y,z)$.
   Decide if they are linear or not, and if linear, whether or not they are homogeneous.
   If nonlinear, decide whether or not they are quasi-linear.

   (a) $aU_{xx} + bU_{yy} = 0$, where $a, b \in \mathbb{R}$ are non-zero.

   (b) $xU_x + yU_y = 0$, where $x, y \in \mathbb{R}$ are non-zero.

   (c) $aUU_{xx} + bU_xU_{yy} = 0$, where $a, b \in \mathbb{R}$ non-zero.

   (d) $\dfrac{\partial^3 U}{\partial^2 x\, \partial y} - \dfrac{\partial U}{\partial y} = x^2 + y^2$

   (e) $x^2U_{yy} - yU_x = U$.

   (f) $x^2U_{yyyy} - yU_x = U^2$.

   (g) $-i\partial_t\psi = \frac{1}{2m}\nabla^2\psi + V(x)\psi$.

   (h) $u_{xx}u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y)$.

   (i) $U_{xx} + y\, U_{yy} = 0$.

   (j) $(\nabla^2)^2\, \Psi := \left[\partial_x^2 + \partial_y^2 + \partial_z^2\right]^2 \Psi = 0$.

2. Find general solutions $U(x,y)$ to the following PDEs (straightforward):

   (a) $\dfrac{\partial U}{\partial x} = x^2 - y^2$.

   (b) $\dfrac{\partial U}{\partial x} - \dfrac{\partial U}{\partial y} = e^x e^{-y}$. [Make an appropriate change of variable.]

   (c) $U_{xx} = \sin y$.

3. Find general solutions $U(x,y)$ to the following PDEs (some mild thinking required):

   (a) $aU_x + bU_y = 0$.

   (b) $U_x\, g_y(x,y) - U_y\, g_x(x,y) = 0$. (Treat $g(x,y)$ as given.)

   (c) $U_{xxyy} = 0$.

   (d) $U_{xx} = y\, U_x + xy$.

4. Eliminate the arbitrary functions from the following and so obtain partial differential equations of which they are the general solution (very straightforward):

   (a) $v = g(x^2 + y^2)$.

   (b) $v = f(x^2 - y^2)$.

   (c) $v = f(x^2 - y^2) + g(x^2 + y^2)$.

   (d) $v = h(2x - y) - g(2x + y)$.

5. **Euler equation: Elliptic/Parabolic/Hyperbolic** (straightforward)

   Determine the Euler type (i.e. elliptic, hyperbolic or parabolic) of each of the following PDEs; and obtain the general solution in each case:

   a. $3U_{xx} + 4U_{xy} - U_{yy} = 0$.

   b. $U_{xx} - 2U_{xy} + U_{yy} = 0$.

   c. $4U_{xx} + U_{yy} = 0$.

   d. $U_{xx} + 4U_{xy} + 4U_{yy} = 0$.

   e. $U_{yy} + 2U_{xx} = 0$.

6. **Euler PDE** (some mild thinking required)

   Starting with the constant-coefficient Euler PDE

$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} = 0,$$

   show that there is a change of independent variables $(x, y) \to (X, Y)$, somewhat different from the change of variables considered in class, such that in terms of the new independent variables

$$U_{XX} + \epsilon\, U_{YY} = 0,$$

   where $\epsilon \in \{-1, 0, +1\}$.

Tutorial exercises — Week 2

1. Determine the order of the following PDEs for a function U, Y, u, or v in terms of x, y or x, t. Decide if they are linear or not, and if so, whether they are homogeneous. If nonlinear, decide whether or not they are quasi-linear.

   (a) $U_t - UU_{xx} + 12xU_x = U$.

   (b) $Y_{xxx} - \cos Y = Y_t$.

   (c) $Y_{xx} + \cos(xy)Y_{yxy} = Y + \ln(x^2 + y^3)$.

   (d) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}$.

   (e) $u_{xy} + \dfrac{\alpha\, u_x - \beta\, u_y}{x - y} = 0$.

   (f) $2u_{tx} + u_x\, u_{xx} - u_{yy} = 0$.

   (g) $u_{xx} + \dfrac{c^2\, y^2}{c^2 - y^2}\, u_{yy} + y\, u_y = 0$.

   (h) $u_t + u_x + uu_x - u_{xxt} = 0$.

   (i) $\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \nu \nabla^2 \vec{v}$.

   (j) $\vec{\nabla} \cdot \vec{v} = 0$.

2. Find general solutions $U(x, y)$ to the following PDEs:

   (a) $\dfrac{\partial U}{\partial y} = \sin xy$.

   (b) $\dfrac{\partial U}{\partial x} + 2\dfrac{\partial U}{\partial y} = 0$. [Make an appropriate change of variable.]

   (c) $U_{xy} = 1$. [Does it matter which order you integrate?]

3. Find general solutions $U(x, y)$ to the following PDEs:

   (a) $U_{xy} = y\, U_x^3$.

   (b) $U_{xy} = xy\, U_y$.

   (c) $U_{xy} = y\, U_y + x^3 y^2$.

   (d) $U_x = U_y$.

4. Eliminate the arbitrary functions from the following and so obtain partial differential equations of which they are the general solution:

   (a) $u = f(x + y)$.

   (b) $u = g(xy)$.

   (c) $u = f(x + y) + g(x - y)$.

   (d) $u = x^n\, h(y/x)$.

Tutorial exercises — Week 3

1. **Euler Equation: Elliptic/Parbolic/Hyperbolic**

   Determine the Euler type (i.e. elliptic, hyperbolic or parabolic) of each of the following PDEs, and obtain the general solution in each case:

   a. $U_{xx} + 4U_{xy} + 4U_{yy} = 0$

   b. $U_{xx} + 2U_{xy} + U_{yy} = 0.$

   c. $U_{xx} + 4U_{yy} = 0.$

   d. $4U_{xx} + 4U_{xy} + 4U_{yy} = 0.$

   e. $U_{yy} + 4U_{xy} + U_{xx} = 0.$

2. Check that *if* $u_1, \ldots, u_m$ are solutions of the heat equation $\sigma^2 u_{xx} = u_t$ for $0 < x < L$ and $t > 0$, with boundary conditions $u(0,t) = u(L,t) = 0$ for all $t > 0$, and $c_1, \ldots, c_m$ are constants, *then* $u = c_1 u_1 + \cdots + c_m u_m$ also satisfies the equation *and* the BCs.