Assignment 2: Partial Differential Equations — PDEs

Separation of Variables and Fourier series

Set: Thursday 28 March 2024 Due: Friday 19 April 2024 at 23:59 (End of week 6)

Read the notes and slides — all this material will be discussed by the end of week 6.

1. **Separation of Variables:**
   Attempt to (partially) solve the following PDEs using separation of variables.

   If possible, determine the resulting ODEs.

   (Do not attempt to actually *solve* the ODEs,
   just find the variable-separated ODEs, if possible.)

   - (a) $xu_{xx} + u_t = 0$;
   - (b) $u_{xx} + (x + t)u_{tt} = 0$.

2. **Separation of Variables:**
   The heat equation for $u(x, y, t)$ in two spatial dimensions has the form

   $$\sigma^2(u_{xx} + u_{yy}) = u_t$$

   If $u(x, y, t) = X(x)Y(y)T(t)$ find ODEs for $X(x)$, $Y(y)$, and $T(t)$.

   (Do not attempt to actually *solve* the ODEs,
   just find the variable-separated ODEs.)

3. **Fourier series:**
   Consider a function $f(x)$ defined on the domain $[-L, +L]$.

   Show that if $f(x) \in C^p$, then there exists some constant K such that the Fourier coefficients satisfy

   $$A_n \leq K/n^p \qquad \text{and} \qquad B_n \leq K/n^p.$$

   Hint #1: Recall the definition of C^p; the function is differentiable at least p times,
   and the p'th derivative is continuous.

   Hint #2: Adapt salient parts of the convergence proof, (Kreyszig's proof for C^2 functions),
   as presented in lectures.

Remember: (Euler formulae)

$$A_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\,dx; \qquad A_{n>0} = \frac{1}{L}\int_{-L}^{+L} f(x)\;\cos(n\pi x/L)\,dx;$$

$$B_0 = 0; \qquad B_{n>0} = \frac{1}{L}\int_{-L}^{+L} f(x)\;\sin(n\pi x/L)\,dx.$$

4. **Fourier series:**
Let $P(x)$ be any arbitrary polynomial of degree m over the interval $[-1, 1]$.

For $n > 0$ set

$$A_n(P) = \int_{-1}^{+1} P(x)\;\cos(n\pi x)\,dx; \qquad B_n(P) = \int_{-1}^{+1} P(x)\;\sin(n\pi x)\,dx.$$

Prove that for $n > 0$ the coefficients $A_n(P)$ and $B_n(P)$ are themselves polynomials, but now in the variable $1/n$, of degree at most m.

Hint #1: Integrate by parts, as many times as needed...

Hint #2: Why does the process of repeated integration by parts eventually stop?

5. **Fourier series for sawtooth function:**
Setup: Consider the (simplified) sawtooth function:

$$\text{(sawtooth function)} = \text{sign}(x) - x \qquad \text{for} \qquad x \in [-1, +1].$$

This is an odd function which is discontinuous at $x = 0$, and zero at $x = \pm 1$, and which can then be extended to a periodic function on the entire real line with period 2.

The Fourier cosine and sine coefficients are then easily determined to be:

$$A_n = 0.$$

$$B_n = \frac{2}{n\pi}.$$

Consider the finite-sum Fourier approximation to the sawtooth function

$$\hat{S}_N(x) = \sum_{n=1}^{N} \frac{2}{n\pi} \sin(n\pi x).$$

See questions overleaf...

Questions:

(a) Sketch the simplified sawtooth function defined above.

(b) Show that:

$$\hat{S}_N(x) = 2\int_0^x \sum_{n=1}^{N} \cos(n\pi u)du.$$

(c) Show that:

$$\sum_{n=1}^{N} \cos(n\pi u) = \frac{1}{2}\left[\frac{\sin([N+\frac{1}{2}]\pi u)}{\sin(\pi u/2)} - 1\right].$$

Hint: Use $e^{i\theta} = \cos\theta + i\sin\theta$, and the well-known series

$$1 + x + x^2 + \cdots + x^m = (1 - x^{m+1})/(1 - x).$$

(d) Show that:

$$\hat{S}_N(x) = \int_0^x \frac{\sin([N+\frac{1}{2}]\pi u)}{\sin(\pi u/2)}du - x.$$

(e) Now suppose $|x| \ll 1$. Then, since $|u| \le |x| \ll 1$, we can approximate $\sin(\pi u/2) \approx \pi u/2$. Show that for $|x| \ll 1$ we now have:

$$\hat{S}_N(x) \approx \frac{2}{\pi}\int_0^x \frac{\sin([N+\frac{1}{2}]\pi u)}{u}du - x.$$

(f) By a suitable change of variables rewrite this as:

$$\hat{S}_N(x) \approx \frac{2}{\pi}\int_0^{[N+\frac{1}{2}]\pi x} \frac{\sin(u)}{u}du - x.$$

(g) Show that this implies

$$\hat{S}_N(x) \approx \frac{2}{\pi}\,\mathrm{Si}\left(\left[N+\frac{1}{2}\right]\pi x\right) - x.$$

(h) Explain the relation between this result for the *sawtooth function*, and the result for the *sign function* that was derived in class:

$$S_M(x) \approx \frac{2}{\pi}\,\mathrm{Si}(2\pi x[M+1]).$$

Tutorial exercises — Week 6

1. Attempt to solve the following PDEs using separation of variables.

   If it is possible, determine the resulting ODEs.

   (a) $tu_{xx} + xu_t = 0$ (b) $u_{xx} + u_{tt} + xu = 0$

2. Using separation of variables, find a solution to Laplace's equation $u_{xx}+u_{yy} = 0$ on the rectangle $0 < x < a$, $0 < y < b$ with Dirichlet boundary conditions:

$$\begin{aligned} &u(0, y) = f(y), \qquad && u(a, y) = g(y), \qquad && 0 < y < b; \\ &u(x, 0) = h(x), && u(x, b) = j(x), && 0 \le x \le a. \end{aligned}$$

   Note all *four* edges are non-zero.
   Hint: Consider adding the solutions to 4 simpler problems.

3. Using separation of variables, find the solution to Laplace's equation in the semi-infinite strip $0 < x < a$, $y > 0$ with boundary conditions

$$\begin{aligned} &u(0, y) = 0, \qquad && y > 0; \\ &u(a, y) = 0, && y > 0; \\ &u(x, 0) = f(x), && 0 \le x \le a; \\ &\lim_{y\to\infty} u(x, y) = 0 && 0 < x < a. \end{aligned}$$

4. Prove that the Fourier coefficients satisfy:

$$|A_0| \le \frac{1}{2L}\int_{-L}^{+L} |f(x)|\ dx; \qquad |A_{n>0}| \le \frac{1}{L}\int_{-L}^{+L} |f(x)|\ dx;$$

$$B_0 = 0; \qquad |B_{n>0}| \le \frac{1}{L}\int_{-L}^{+L} |f(x)|\ dx.$$

   (Much stronger results are actually known.)

   Hint: Remember:

$$A_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\ dx; \qquad A_{n>0} = \frac{1}{L}\int_{-L}^{+L} f(x)\ \cos(n\pi x/L)\ dx;$$

$$B_0 = 0; \qquad B_{n>0} = \frac{1}{L}\int_{-L}^{+L} f(x)\ \sin(n\pi x/L)\ dx.$$

5. Consider the finite sum:

$$S_M(x) = \frac{4}{\pi}\left\{\sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots + \frac{\sin([2M+1]\pi x)}{2M+1}\right\},$$

   which we saw is of interest in analyzing the Gibbs phenomenon for step functions.

(a) Show that:

$$S_M(x) = 4\int_0^x \{\cos(\pi u) + \cos(3\pi u) + \cos(5\pi u) + \cdots + \cos([2M+1]\pi u)\}\, du.$$

(b) Show that:

$$\cos(\pi u) + \cos(3\pi u) + \cos(5\pi u) + \cdots + \cos([2M+1]\pi u) = \frac{\sin([2M+2]\pi u)}{2\sin(\pi u)}$$

Hint: This is "merely" a trig identity.
Hint: Use $e^{i\theta} = \cos\theta + i\sin\theta$, and the well-known series
$$1 + x + x^2 + \cdots + x^m = (1 - x^{m+1})/(1 - x).$$

(c) Show that:

$$S_M(x) = 2\int_0^x \frac{\sin([2M+2]\pi u)}{\sin(\pi u)} du.$$

(d) Show that:

$$S_M\left(\frac{x}{2M+2}\right) = 2\int_0^x \frac{\sin(\pi u)}{\sin(\pi u/[2M+2])} \frac{du}{2M+2}.$$

(e) Show that:

$$\lim_{M\to\infty} S_M\left(\frac{x}{2M+2}\right) = \frac{2}{\pi}\int_0^x \frac{\sin(\pi u)}{u} du = \frac{2}{\pi}\int_0^{\pi x} \frac{\sin(u)}{u} du = \frac{2}{\pi}\,\mathrm{Si}(\pi x).$$

This is another way of getting to the key result for the (step-function) Gibbs phenomenon. (It is very closely related, *but not identical to*, question 5 of the assignment.)
