VICTORIA UNIVERSITY OF WELLINGTON
SCHOOL OF MATHEMATICS AND STATISTICS
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MATH 301 — PARTIAL DIFFERENTIAL EQUATIONS — 2024

In-term test # 1:

Tuesday 26 March 2024; 12:00-12:50; Murphy 220

FAMILY NAME	PERSONAL NAME(S)	STUDENT NUMBER

Most questions should be very easy; some questions might require a little thinking.

1. **Classification of PDEs:** **[10 marks out of 100]**
For the following list of PDEs:
   - State the order.
   - State whether or not the PDE is linear. (Yes/No.)
   - State whether or not the PDE is quasi-linear. (Yes/No/Automatic.)
   - State whether or not it is homogeneous. (Yes/No/Meaningless.)

   (a) $\det\left[\partial_i \partial_j U(x^k)\right] = 0$.

2^{nd}-order	nonlinear	not quasi-linear	meaningless

   (b) $U_{xxxx} + 2U_{xyxy} + U_{yyyy} = 0$.

4^{th}-order	linear	automatic	homogeneous

   (c) $\det\left[\delta_{ij} + U(x^k)\,\partial_i U(u^k)\,\partial_j U(x^k)\right] = S(x^k)$.

1^{st}-order	nonlinear	not quasi-linear	meaningless

   (d) $\exp(a\,\partial_x + b\,\partial_y)\,U(x,y) = U(x,y)$.

infinite order	linear	automatic	homogeneous

   (e) $f(U)U_x + g(U)U_y = 0$.

1^{st}-order	nonlinear	quasilinear	meaningless

2. **General solutions:** **[10 marks out of 100]**

Find the general solution (you may need to think just a little bit) to the following PDEs:

(a) $U_{xyz} = 0$. (Straightforward.)

By inspection:

$$U(x,y,z) = f(x,y) + g(y,z) + h(z,x).$$

Systematic:

$$U_{xyz} = 0 \quad \Longrightarrow \quad U_{xy} = f_1(x,y) \quad \Longrightarrow \quad U_x = \int f_1(x,y)dy + f_2(x,z) = f_3(x,y) + f_2(x,z)$$

But then

$$U = \int [f_3(x,y) + f_2(x,z)]dx + f_4(y,z) = f_5(x,y) + f_6(x,z) + f_4(y,z)$$

Re-name the arbitrary funcrtions

$$U(x,y,z) = f(x,y) + g(y,z) + h(z,x).$$

(b) $\exp\left(a\,\partial_x + b\,\partial_y\right)\; U(x,y) = U(x,y).$ (This one is tricky).

Hint: Evaluate $\exp\left(a\,\frac{\mathrm{d}}{\mathrm{d}x}\right) W(x)$ using Taylor series; draw the obvious conclusion.

Use the hint:

$$\exp\left(a\,\frac{\mathrm{d}}{\mathrm{d}x}\right) W(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(a\,\frac{\mathrm{d}}{\mathrm{d}x}\right)^n W(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n W(x) = \sum_{n=0}^{\infty} \frac{1}{n!} a^n W^{(n)}(x)$$

But that last expression is just the Taylor theorem expansion for $W(x+a)$. That is

$$\exp\left(a\,\frac{\mathrm{d}}{\mathrm{d}x}\right) W(x) = W(x+a).$$

Thence the PDE of interest

$$\exp\left(a\,\partial_x + b\,\partial_y\right)\; U(x,y) = U(x,y)$$

has the general solution

$$U(x+a, y+b) = U(x,y).$$

(That is, the field $U(x,y)$ is periodic in both x and y.)

3. **From general solution to PDE:** **[10 marks out of 100]**

Suppose the functions presented below are the general solution to *some* PDE.
Write down the relevant PDE:

(a) $U(x, y) = f(ax + by)$.

By the chain rule

$$U_x = af'(ax + by); \qquad U_y = bf'(ax + by)$$

Thence

$$bU_x = aU_y.$$

(Or any equivalent form.)

(b) $U(x, t) = f(x^a t^b)$.

By the chain rule

$$U_x = ax^{a-1}t^b f'(x^a t^b); \qquad U_t = bx^a t^{b-1} f'(x^a t^b)$$

Thence

$$bxU_x = atU_t.$$

(Or any equivalent form.)

4. **Frobenius–Mayer systems:** **[10 marks out of 100]**
Consider, in 3 space dimensions, the special Frobenius–Mayer system:

$$\vec{\nabla}\Phi = \vec{v}(\vec{x}, \Phi).$$

(a) Write down the corresponding Frobenius integrability conditions.
(Try to make them look as simple as possible.)

To match the notation of the lectures write this Frobenius–Mayer system as

$$\partial_i \Phi(x^m) = v_i(x^m, \Phi(x^m))$$

Evaluate the 2nd partial derivative:

$$\partial_j \partial_i \Phi(x^m) = \partial_j v_i(x^m, \Phi(x^m)) + \frac{\partial v_i(x^m, \Phi(x^m))}{\partial \Phi(x^m)} \partial_j \Phi(x^m)$$

Use the PDE:

$$\partial_j \partial_i \Phi(x^m) = \partial_j v_i(x^m, \Phi(x^m)) + \frac{\partial v_i(x^m, \Phi(x^m))}{\partial \Phi(x^m)} v_j(x^m, \Phi(x^m))$$

But partial derivatives commute, so (supressing arguments for clarity)

$$\partial_i v_j - \partial_j v_i + \frac{\partial v_i}{\partial \Phi} v_j - \frac{\partial v_j}{\partial \Phi} v_i = 0$$

Recaognize that these are just the curl and a vector cross product:

$$(\nabla \times \vec{v}) + \left(\frac{\partial \vec{v}}{\partial \Phi} \times \vec{v} \right) = 0.$$

(b) How many integrability conditions are there?
(Careful!)

Three.

5. **Very simple Euler equations:** **[10 marks out of 100]**

(a) Write down the general solution for the PDE $U_{xx} + U_{yy} = 0$.

By inspection
$$U(x, y) = f(x + iy) + g(x - iy).$$
(I told you to memorize this.)

(b) Write down the general solution for the PDE $c^2\, U_{xx} - U_{tt} = 0$.

By inspection
$$U(x, t) = f(x + ct) + g(x - ct).$$
(I told you to memorize this.)

6. **Euler type:** **[10 marks out of 100]**
Write down the Euler type (elliptic, parabolic, hyperbolic) for each of the following PDEs.

(a) $U_{xx} + 2U_{xy} + U_{yy} = 0.$

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0 \quad \Longrightarrow \quad \text{Parabolic.}$$

(b) $U_{xx} + 4U_{xy} + U_{yy} = 0.$

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3 < 0 \quad \Longrightarrow \quad \text{Hyperbolic.}$$

(c) $U_{xx} + U_{xy} + U_{yy} = 0.$

$$\det \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \frac{3}{4} > 0. \quad \Longrightarrow \quad \text{Elliptic.}$$

7. **D'Alambert's solution:** **[15 marks out of 100]**

Consider the function:

$$\boxed{U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ \mathrm{d}s.}$$

(a) Evaluate $U(x,0)$:

$$U(x,0) = \frac{1}{2}[f(x+0) + f(x-0)] + \frac{1}{2c}\int_{x-0}^{x+0} g(s)\ \mathrm{d}s = f(x).$$

(b) Evaluate $U_t(x,0)$:

$$U_t(x,t) = \partial_t \left[\frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ \mathrm{d}s\right]$$

Chain rule

$$U_t(x,t) = \left[\frac{1}{2}c[f'(x+ct) - f'(x-ct)] + \frac{1}{2c}c[g(x+ct) + g(x-ct)\right]$$

$$U_t(x,0) = \left[\frac{1}{2}c[f'(x+0) - f'(x-0)] + \frac{1}{2c}c[g(x+0) + g(x-0)\right] = g(x)$$

That is

$$U_t(x,0) = g(x).$$

(c) Find (with a sketch proof) a differential equation that $U(x,t)$ satisfies:

Re-write

$$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ \mathrm{d}s$$

as

$$U(x,t) = \left(\frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{0}^{x+ct} g(s)\ \mathrm{d}s\right) + \left(\frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} g(s)\ \mathrm{d}s\right)$$

That is

$$U(x,t) = F(x+ct) + G(x-ct)$$

This makes it obvious that $U(x,t)$ satisfies the wave equation.

(Or you could just brute force evaluate $U_{xx}(x,t)$ and $U_{tt}(x,t)$.)

8. **Euler equation (variable coefficients):** **[10 marks out of 100]**
What is special about the generalized variable-coefficient Euler PDE in 2 dimensions?

$$\boxed{a(x,y)\,U_{xx} + 2h(x,y)\,U_{xy} + b(x,y)\,U_{yy} = F(x,y,U,U_x,U_y).}$$

Give a complete list of the four simplified PDEs, (both the equations and the names), that can be obtained from this general form of the PDE by suitable changes of the coordinates.

One option: $$U_{\bar{x}\bar{x}} - U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x},\bar{y},U,U_{\bar{x}},U_{\bar{y}}).$$ Wave equation with source.
One option: $$U_{\bar{x}\bar{x}} + U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x},\bar{y},U,U_{\bar{x}},U_{\bar{y}}).$$ Laplace equation with source.
One option: $$U_{\bar{x}\bar{x}} = \tilde{F}(\bar{x},\bar{y},U,U_{\bar{x}},U_{\bar{y}}).$$ Parabolic equation with source.
One option: $$U_{\bar{x}\bar{x}} - \bar{x}U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x},\bar{y},U,U_{\bar{x}},U_{\bar{y}}).$$ Tricomi equation with source.

9. **Separation of variables:** **[15 marks out of 100]**

Consider the so-called "telegrapher's equation":

$$\boxed{u_{tt} = u_{xx} + \sigma u_x.}$$

(This is used as a model for a wave equation with damping/friction.)

(The field $u(x,t)$, *and the separation constant*, are most usefully taken to be *complex*.)

(a) Separate variables using the ansatz $u(x,t) = X(x)\,T(t)$.
What ODEs do $X(x)$ and $T(t)$ satisfy?
Use K for the separation constant,
(we will want to use the symbol k for other purposes).

Note:

$(X(x)T(t))_{tt} = T''(t)X'(x)$; $(X(x)T(t))_{xx} = T(t)X''(x)$; $(X(x)T(t))_x = T(t)X'(x)$,

The PDE becomes

$$T''X = TX'' + \sigma TX$$

Divide by XT

$$\frac{T''}{T} = \frac{X'' + \sigma X'}{X} = K$$

Two separated ODEs:

$$T'' = KT; \qquad X'' + \sigma X' = KX.$$

(b) Solve the ODEs for $X(x)$ and $T(t)$. Do not impose any boundary conditions.
Hint: Use exponentials of complex numbers.
Try $T(t) = \exp(i\omega t)$ and $X(x) = \exp(-ikx)$.
Find $K(\omega)$ and $K(k)$.
That is, evaluate the separation constant K in terms of the parameters ω and k.

$$K(\omega) = \frac{T''}{T} = \frac{(e^{i\omega t})''}{e^{i\omega t}} = -\omega^2.$$

$$K(k) = \frac{X'' + \sigma X'}{X} = \frac{(e^{-ikx})'' + \sigma(e^{-ikx})'}{e^{-ikx}} = -k^2 - ik\sigma = -k(k + i\sigma).$$

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