Tutorial for week 2 — Euler PDEs

Set: Monday 4 March 2024 Due: Thursday 14 March 2024

1. **Euler Equation: Elliptic/Parbolic/Hyperbolic**

   Determine the Euler type (i.e. elliptic, hyperbolic or parabolic) of each of the following PDEs, and obtain the general solution in each case:

   a. $U_{xx} + 4U_{xy} + 4U_{yy} = 0$

   Solution:

   $\det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$. Therefore parabolic.

   Find roots of $1 + 4z + 4z^2 = 0$, that is $(1 + 2z)^2 = 0$.

   Repeated root $z = -\frac{1}{2}$.

   General solution is $U = F(x - \frac{1}{2}y) + (x + c)\, G(x - \frac{1}{2}y)$ for any $c \neq -\frac{1}{2}$.

   Might as well take $c = 0$ so that $U = F(2x - y) + x\, G(2x - y)$.

   Can always check by differentiating.

   b. $U_{xx} + 2U_{xy} + U_{yy} = 0$.

   Solution:

   $\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$. Therefore parabolic.

   Find roots of $1 + 2z + z^2 = 0$, that is $(1 + z)^2 = 0$.

   Repeated root $z = -1$.

   General solution is $U = F(x - y) + (x + c)\, G(x - y)$ for any $c \neq -1$.

   Might as well take $c = 0$ so that $U = F(x - y) + x\, G(x - y)$.

   Can always check by differentiating.

   c. $U_{xx} + 4U_{yy} = 0$.

   Solution:

   $\det \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = 4 > 0$. Therefore elliptic.

   Find roots of $1 + 0z + 4z^2 = 0$, that is $4z^2 = -1$.

   Distinct complex roots $z = \pm\frac{i}{2}$.

   General solution is $U = F(x + \frac{i}{2}y) + G(x - \frac{i}{2}y)$.

   Might as well rescale and take $U = F(2x + iy) + G(2x - iy)$.

   Can always check by differentiating.

d. $4U_{xx} + 4U_{xy} + 4U_{yy} = 0$.

Solution:

$\det \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = 12 > 0$. Therefore elliptic.

Find roots of $4 + 4z + 4z^2 = 0$, that is $1 + z + z^2 = 0$.

Distinct complex roots $z = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$.

General solution is $U = F(x + [-\frac{1}{2} + \frac{i\sqrt{3}}{2}]y) + G(x + [-\frac{1}{2} - \frac{i\sqrt{3}}{2}]y)$.

Might as well rescale and take $U = F(2x + [-1 + i\sqrt{3}]y) + G(2x + [-1 - i\sqrt{3}]y)$.

Can always check by differentiating.

e. $U_{yy} + 4U_{xy} + U_{xx} = 0$.

Solution:

$\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3 < 0$. Therefore hyperbolic.

Find roots of $1 + 4z + z^2 = 0$.

Distinct real roots $z = -2 \pm \sqrt{3}$.

General solution is $U = F(x + [-2 + \sqrt{3}]y) + G(x + [-2 - \sqrt{3}]y)$.

Can always check by differentiating.

2. Check that *if* $u_1, \ldots, u_m$ are solutions of the heat equation $\sigma^2 u_{xx} = u_t$ for $0 < x < L$ and $t > 0$, with boundary conditions $u(0,t) = u(L,t) = 0$ for all $t > 0$, and $c_1, \ldots, c_m$ are constants, *then* $u = c_1 u_1 + \cdots + c_m u_m$ also satisfies the equation *and* the BCs.

Solution:

Pretty much obvious.

The heat equation is 2^{nd} order *linear* and *homogeneous*.

Therefore, if the individual functions u_i satisfy the heat equation,
then so does the arbitrary linear combination $u = c_1 u_1 + \cdots + c_m u_m$.

Furthermore, note that the quoted boundary conditions are also *linear* and *homogeneous*.

QED.
