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MATH 301 | Differential equations | 2024

Tutorial Week 2: Partial Differential Equations — PDEs

Set: Monday 4 March 2024 | Due: Thursday 7 March 2024

1. Determine the order of the following PDEs for a function U, Y, u, or v in terms of x, y or x, t. Decide if they are linear or not, and if so, whether they are homogeneous.
If nonlinear, decide whether or not they are quasi-linear.

   (a) $U_t - UU_{xx} + 12xU_x = U$.
   Solution: 2^{nd} order; non-linear; (homogeneity is meaningless); quasi-linear.

   (b) $Y_{xxx} - \cos Y = Y_t$.
   Solution: 3^{rd} order; nonlinear; (homogeneity is meaningless); quasi-linear.

   (c) $Y_{xx} + \cos(xy)Y_{yxy} = Y + \ln(x^2 + y^3)$.
   Solution: 3^{rd} order; linear; non-homogeneous; (automatically quasi-linear).

   (d) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}$.
   Solution: 4^{th} order; linear; homogeneous; (automatically quasi-linear).

   (e) $u_{xy} + \dfrac{\alpha\, u_x - \beta\, u_y}{x - y} = 0$.
   Solution: 2^{nd} order; linear; homogeneous; (automatically quasi-linear).

   (f) $2u_{tx} + u_x\, u_{xx} - u_{yy} = 0$.
   Solution: 2^{nd} order; non-linear; (homogeneity is meaningless); quasi-linear.

   (g) $u_{xx} + \dfrac{c^2\, y^2}{c^2 - y^2}\, u_{yy} + y\, u_y = 0$.
   Solution: 2^{nd} order; linear; homogeneous; (automatically quasi-linear).

   (h) $u_t + u_x + uu_x - u_{xxt} = 0$.
   Solution: 3^{rd} order; non-linear; (homogeneity is meaningless); quasi-linear.

   (i) $\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \nu \nabla^2 \vec{v}$.
   Solution: 2^{nd} order; non-linear; (homogeneity is meaningless); quasi-linear.

   (j) $\vec{\nabla} \cdot \vec{v} = 0$.
   Solution: 1^{st} order; linear; homogeneous; (automatically quasi-linear).

2. Find general solutions $U(x, y)$ to the following PDEs:

   (a) $\dfrac{\partial U}{\partial y} = \sin xy$.
   Solution:
   $$U(x, y) = \int \sin(xy)dy + F(x).$$
   More specifically
   $$U(x, y) = \int_0^y \sin(x\bar{y})d\bar{y} + F(x) = \frac{1 - \cos(xy)}{x} + F(x) = -\frac{\cos(xy)}{x} + \tilde{F}(x)$$

   (b) $\dfrac{\partial U}{\partial x} + 2\dfrac{\partial U}{\partial y} = 0$. [Make an appropriate change of variable.]
   Solution:
   (Devious): Write this as $\vec{n} \cdot \nabla U = 0$ wuth $\vec{n} = (1, 2)$.
   Note this means $(1, 2) \perp (U_x, U_y)$.
   Thence $(2, -1) \parallel (U_x, U_y)$ and so $U(x, y) = F(2x - y)$.

   (c) $U_{xy} = 1$. [Does it matter which order you integrate?]
   Solution:
   First $U_x = \int 1dy + f(x) = y + f(x)$.
   Then $U = \int U_x dx = \int (y + f(x))dx + G(y) = xy + F(x) + G(y)$.
   The order in which you integrate does not matter.

3. Find general solutions $U(x, y)$ to the following PDEs:

   (a) $U_{xy} = y\, U_x^3$.
   Solution:
   $$\frac{\partial_y(U_x)}{U_x^3} = y; \qquad -\frac{1}{2U_x^2} = \frac{y^2}{2} - \frac{f(x)}{2}; \qquad U_x^2 = \frac{1}{f(x) - y^2}.$$
   Thence
   $$U(x, y) = \int \frac{dx}{\sqrt{f(x) - y^2}} + G(y).$$

   (b) $U_{xy} = xy\, U_y$.
   Solution:
   $$\frac{\partial_x(U_y)}{U_y} = xy; \qquad \ln(U_y) = \int yx dx + F(y) = \frac{x^2 y}{2} + F(y); \qquad U_y = e^{F(y)} e^{x^2 y/2}.$$
   $$U(x, y) = \int e^{F(y)} e^{\frac{1}{2}x^2 y} dy + G(x) = \int \tilde{F}(y) e^{\frac{1}{2}x^2 y} dy + G(x).$$
   Not much more can be done...

(c) $U_{xy} = y\, U_y + x^3 y^2$.

Solution:

First write the PDE as

$$\partial_x(U_y) = y\, U_y + x^3 y^2.$$

This is 1st order linear ODE for U_y.

So you should try to guess an IF (integrating factor):

$$\partial_x(e^{-xy} U_y) = e^{-xy} x^3 y^2.$$

Then

$$(e^{-xy} U_y) = \int e^{-xy} x^3 y^2 dx + f(y).$$

$$U_y = e^{xy} \left\{ y^2 \int e^{-xy} x^3 dx + f(y) \right\}.$$

$$U = \int y^2 e^{xy} \left[\int e^{-xy} x^3 dx \right] dy + \int e^{xy} f(y) dy + G(x).$$

This is (at least formally) the full answer...

To finish the job, integrate by parts (boring):

$$U = -\frac{x^3 y^2}{2} - 3x^2 y - 6x \ln y + \frac{6}{y} + \int e^{xy} f(y) dy + G(x).$$

(d) $U_x = U_y$.

Solution:

Note $(1, -1) \perp (U_x, U_y)$, so $(1, 1) \parallel (U_x, U_y)$, so $U(x, y) = f(x + y)$.

4. Eliminate the arbitrary functions from the following and so obtain partial differential equations of which they are the general solution:

(a) $u = f(x + y)$.

Solution: We have

- $u_x = f'(x + y)$
- $u_y = f'(x + y)$
- Therefore $u_x = u_y$, which can also be written as $u_x - u_y = 0$.

(b) $u = g(xy)$.

Solution: We have

- $u_x = yg'(xy)$
- $u_y = xg'(xy)$
- Therefore $xu_x = yu_y$, which can also be written as $xu_x - yu_y = 0$.

(c) $u = f(x+y) + g(x-y)$.

Solution: We have

- $u_x = f'(x+y) + g'(x-y)$
- $u_y = f'(x+y) - g'(x-y)$
- $u_{xx} = f''(x+y) + g''(x-y)$
- $u_{yy} = f''(x+y) + g''(x-y)$
- Therefore $u_{xx} = u_{yy}$, which can also be written as $u_{xx} - u_{yy} = 0$. (The wave equation.) (It turns out we do not need to know u_{xy}, but this is a bit of luck in this specific case...)

(d) $u = x^n\, h(y/x)$.

Solution:

- $u_x = nx^{n-1}h(y/x) - yx^{n-2}h'(y/x) = nu/x - yx^{n-2}h'(y/x)$
- $u_y = x^{n-1}h'(y/x)$
- Therefore $u_x + (y/x)u_y = nu/x$, which can also be written as $xu_x + yu_y = nu$.
- Note that the differential equation is symmetric under the interchange of $x \leftrightarrow y$, but the general solution does not seem to have this symmetry? What is going on here?
- Rewrite $u = x^n\, h(y/x) = (xy)^{n/2}(x/y)^{n/2}h(x/y) = (xy)^{n/2}\, \tilde{h}(x/y)$.
  This makes the symmetry a little more obvious.
