MATH301: Differential Equations
Lesson 1: The transport equation

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Traveling waves

Definition: Traveling wave

A wave that travels without change in shape and speed. In abundance in nature:

- The light (in vacuum)
- Water waves in a straight channel with flat bottom
- Signals in optical fibers
- Waves in ultra-cold matter, plasma waves and more

Formula

A traveling wave that travels with constant speed c

- Changes location x with time t: Thus $u(t,x)$
- At $t=0$ has shape $f(x)$: Thus $u(0,x)=f(x)$
- Travels with constant speed c thus $u(t,x)=f(x-ct)$

The transport equation: The simplest wave equation

Take $u(t,x) = f(k)$, where $k = x - ct$ The,

$$\begin{aligned} u_t(t,x) &= \frac{\partial}{\partial t} u(t,x) \\ &= \frac{\partial}{\partial t} f(k) \\ \text{Chain rule} \quad &= \frac{\partial k}{\partial t} \frac{\partial}{\partial k} f(k) \\ &= -c f'(k) \end{aligned}$$

Because if you see $k = k(t,x)$ then

$$\frac{\partial k}{\partial t} = -c$$

The transport equation

Similarly $u(t,x) = f(k)$, where $k = x - ct$ The,

$$\begin{aligned} u_x(t,x) &= \frac{\partial}{\partial x} u(t,x) \\ &= \frac{\partial}{\partial x} f(k) \\ \text{Chain rule } &= \frac{\partial k}{\partial x}\frac{\partial}{\partial k} f(k) \\ &= f'(k) \end{aligned}$$

Because if you see $k = k(t,x)$ then

$$\frac{\partial k}{\partial x} = 1$$

The transport equation

Thus

$$u_t(t,x) = -cf'(k) = -cu_x(t,x) \ ,$$

which means that any traveling wave satisfies the equation

$$u_t + cu_x = 0$$

This is known as the (uniform) transport equation.

The transport equation

Initial value problem

Solve the Initial Value Problem

$$u_t + 2u_x = 0 \ ,$$

when

$$u(0,x) = e^{-x^2} \ .$$

The solution is $u(t,x) = e^{-(x-2t)^2}$.

The transport equation

- The lines $x = ct + k$ are called characteristic lines and are all parallel
- The $x - t$ diagram is called space-time diagram
- The solution $u(t, x) = f(x - ct)$ is constant along the lines $x - ct = k$, $k \in \mathbb{R}$
- The line passes starts at the point $(0, k)$ (at $t = 0$ we have $x = k$).

General methodology for solving $u_t + cu_x = 0$

- Define the characteristic line to be the line with slope c (whatever is in front of u_x)
- This means that $x'(t) = c$. Let the line also passing from $x(0) = k$ at $t = 0$.
- Integrating we find $x(t) = ct + k$ the characteristic line
- We observe that

$$\frac{d}{dt}u(t, x(t)) = u_t(t, x(t)) + cu_x(t, x(t)) = 0$$

  and thus the solution is constant along the characteristic line $x(t) = ct + k$

$$u(t, x(t)) = \text{constant}$$

Why we need more wave equations?

- Transport equation is linear while waves in nature are nonlinear with small exceptions
- Waves are dispersive, i.e. waves of different height/wavelength propagate with different speed
- There are other phenomena such as decay and wave breaking we want to describe

Transport with Decay

Let $c > 0$, $a > 0$ constants and $u = u(t, x)$. We define the transport equation with decay as

$$u_t + cu_x + au = 0$$

Why? - Multiply with u both sides and integrate from $-\infty$ to $+\infty$ in space:

$$\int_{-\infty}^{+\infty} u_t u \; dx + c \int_{-\infty}^{+\infty} u_x u \; dx = -a \int_{-\infty}^{+\infty} u^2 \; dx < 0$$

The second integral is 0:

$$\int_{-\infty}^{+\infty} u_x u \, dx = \frac{1}{2} \int_{-\infty}^{+\infty} (u^2)_x \, dx = [u^2]_{-\infty}^{+\infty} = u^2(+\infty) - u^2(-\infty) = 0 - 0 \, .$$

Thus, we get

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} u^2 \; dx = -a \int_{-\infty}^{+\infty} u^2 \; dx < 0$$

Transport with Decay

The function

$$E(t) = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} u^2 \; dx$$

is the total energy of the wave (here) and is decreasing (decaying). The larger the value of a the larger the decay it is.

Transport with Decay: Solution

Let

$$u_t + cu_x + au = 0$$

- We set $u(t,x) = v(t,k)$ where $k = x - ct$ as usual.
- We get $v_t + av = 0$ which is a first-order ODE
- We multiply with the integrating factor e^{at} and get

  $$(e^{at}v)_t = 0$$

  and thus

  $$v(t,k) = f(k)e^{-at}$$

- Back to the physical coordinates

  $$u(t,x) = f(x-ct)e^{-at}$$