MATH301: Differential Equations
Partial Differential Equations

Dimitrios Mitsotakis
dimitrios.mitsotakis@vuw.ac.nz

School of Mathematics and Statistics
Victoria University of Wellington

Nonlinear diffusion

Consider the Burger's equation

$$u_t + uu_x = \gamma u_{xx}$$

$\gamma > 0$ is the diffusion coefficient.

- This equations does not have shock waves
- It has regularised shock waves as traveling waves
- dissipates the energy

Multiply Burger's equation with u and integrate to obtain

$$E'(u) = \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{+\infty} u^2\ dx = -\gamma \int_{-\infty}^{+\infty} u_x^2\ dx < 0$$

This means that the energy $E(u)$ is decreasing.

traveling waves

Take

$$u(t,x) = v(x-ct)$$

Introduce new independent variable

$$\xi = x - ct$$

Then

$$u_t = \frac{d}{dt}u(t,x) = \frac{d\xi}{dt}\frac{d}{d\xi}v(\xi) = -cv'$$

similarly

$$u_x = \frac{d}{dx}u(t,x) = \frac{d\xi}{dx}\frac{d}{d\xi}v(\xi) = v'$$

and

$$u_{xx} = v''$$

traveling waves

Substitution into the Burger's equation implies

$$-cv' + vv' = \gamma v''$$

Integration with respect to ξ gives

$$-cv + \frac{1}{2}v^2 + k = \gamma v'$$

where k is the integration constant.
Physical solutions have $v'(\xi) \to 0$ as $|\xi| \to \infty$ otherwise the solutions will have infinite energy! Thus taking $\xi \to \pi\infty$ to the first order ODE we get that

$$\frac{1}{2}v_{\pm\infty}^2 - cv_{\pm\infty} + k = 0$$

where

$$\lim_{\xi \to \pm\infty} v(\xi) = v_{\pm\infty}$$

traveling waves

Solving the quadratic equation

$$\frac{1}{2}v^2 - cv + k = 0$$

we obtain that for $k < \frac{c^2}{2}$ we have two real roots

$$v_{\pm\infty} = c^2 \pm \sqrt{c^2 - 2k}$$

Following the book of Olver we call $v_{+\infty} = a$ and $v_{-\infty} = b$. Then $c = \frac{1}{2}(a+b)$ and $k = \frac{1}{2}ab$. Then we write the 1st order ODE as

$$2\gamma \frac{dv}{d\xi} = (v-a)(v-b)$$

traveling waves

$$2\gamma \frac{dv}{d\xi} = (v-a)(v-b)$$

For bounded solutions we must require $a < v < b$ (we will see it in the next lines but also we want $v' < 0$)
We write the ODE as a separable ODE

$$\int \frac{2\gamma dv}{(v-a)(v-b)} = \int d\xi$$

The first integral gives

$$\int \frac{2\gamma dv}{(v-a)(v-b)} = \frac{2\gamma}{b-a} \log\left(\frac{b-v}{v-a}\right)$$

and thus

$$\frac{2\gamma}{b-a} \log\left(\frac{b-v}{v-a}\right) = \xi + \delta$$

where δ new integration constant.

traveling waves

Keeping δ arbitrary we solve for

$$v(\xi) = \frac{ae^{(b-a)(\xi-\delta)/(2\gamma)} + b}{e^{(b-a)(\xi-\delta)/(2\gamma)} + 1} ,$$

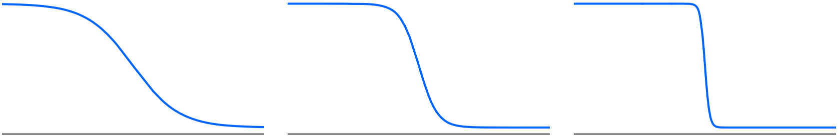
and thus

$$u(t,x) = \frac{ae^{(b-a)(x-ct-\delta)/(2\gamma)} + b}{e^{(b-a)(x-ct-\delta)/(2\gamma)} + 1} .$$

Uniqueness

The traveling wave is unique (except for horizontal translations) meaning that for each $c > 0$ there is only one traveling wave (perhaps translated horizontally due to $-\delta$).

traveling waves



$\gamma = .25$ $\gamma = .1$ $\gamma = .025$

For smaller values of γ the traveling wave becomes steeper but it will always remain smooth (differentiable)

Dispersive waves

For waves ω is the frequency, k the wave number then the phase (actual) speed is

$$c = \frac{\omega}{k}$$

This means that if

$$\frac{d^2\omega}{dk^2} \neq 0$$

then waves with different wavelength ($\lambda = 2\pi/k$) travel with different speed.

Dispersive waves

Dispersive waves are waves that their speed depends on the wavelength (wavenumber or amplitude).

Dispersive waves

Consider the linearised KDV equation

$$u_t + u_{xxx} = 0$$

and search for solutions of the form

$$u(t,x) = Ae^{i(kx-\omega t)}$$

These are periodic, traveling waves (for example take the real part to get a sinus function).
Substitution into the LKDV we obtain

$$-i\omega + (ik)^3 = 0$$

or better

$$\omega(k) = k^3$$

The relationship between ω and k is called (linear) dispersion relationship. We can only estimate for linear waves.

Solitary waves - Solitons

Consider the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

We will search for traveling waves of the form

$$u(t,x) = v(\xi) = v(x - ct)$$

where $\xi = x - ct$ for any $c > 0$. By chain rule we have

$$u_t = -cv', \qquad u_x = v', \qquad u_{xxx} = v'''$$

and thus we get

$$v''' + vv' - cv' = 0$$

Solitary waves - Solitons

Because solitary waves as described by J-S Russel have $u \to 0$ along with their derivatives as $|\xi| \to \pm\infty$, we get after integration

$$v'' + \frac{1}{2}v^2 - cv = 0$$

We multiply with v' and integrate again to obtain

$$\frac{1}{2}(v')^2 + \frac{1}{6}v^3 - \frac{1}{2}cv^2 = 0$$

Solving for v' we get

$$\frac{dv}{d\xi} = v\sqrt{c - \frac{1}{3}v}$$

Solitary waves - Solitons

$$\frac{dv}{d\xi} = v\sqrt{c - \frac{1}{3}v}$$

Set

$$w^2 = c - \frac{1}{3}v$$

we get $v' = -6ww'$ and thus

$$-6ww' = 3(c - w^2)w$$

or better

$$\frac{w'}{w^2 - c} = \frac{1}{2}$$

for $\xi < 0$

Solitary waves - Solitons

$$\frac{w'}{w^2 - c} = \frac{1}{2}$$

Integration (simple fractions) gives

$$\log\left(\frac{\sqrt{c}+w}{\sqrt{c}-w}\right) = \frac{1}{2}\left[\sqrt{c}\xi + \delta\right]$$

Solving for w we obtain

$$w = \sqrt{c}\frac{e^{\frac{1}{2}\left[\sqrt{c}\xi+\delta\right]} - 1}{e^{\frac{1}{2}\left[\sqrt{c}\xi+\delta\right]} + 1} = \sqrt{c}\,\tanh\left(\frac{1}{2}\left[\sqrt{c}\xi + \delta\right]\right)$$

Since $v = 3(c - w^2)$ and $\operatorname{sech}^2 = 1 - \tanh^2$ we have

$$u(t,x) = v(x - ct) = 3c\,\operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}(x - ct) + \delta\right]$$