Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tiessday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Introduction

Introduction:

- This course, [Math 301, PDEs], is an introduction to Partial Differential Equations.
- PDEs are one of the most useful tools of applied mathematics and mathematical physics. If you intend to continue studying in either of these fields, get used to working with PDEs — they are ubiquitous.
- PDEs are also central to mathematical finance, where they underlie (for instance) the Black–Scholes theory for the pricing of stock market options and [financial] derivatives.

Reference materials

Introduction:

- Modern textbook (for background reference):
  Peter J Olver,
  Introduction to Partial Differential Equations,
  Springer.
- Part 1 lectures correspond very roughly to Chapters 1 to 4 of Olver.
- Traditional textbook:
  Boyce and DiPrima,
  Elementary Differential Equations and Boundary Value Problems,
- Part 1 lectures correspond very roughly to Chapter 10 of
  Boyce and DiPrima: Partial differential equations and Fourier series.
- Note:
  ODE = Ordinary Differential Equation;
  PDE = Partial Differential Equation.

Introduction:

In addition to the "official" textbook,and this set of notes, there are many other books you can look at for additional background material, ideas, and examples.

Good solid textbooks include:

- Erwin Kreyszig,
  "Advanced engineering mathematics".
- Stanley Farlow,
  "Partial differential equations for scientists and engineers".
- Ronald Guenther and John Lee,
  "Partial differential equations of mathematical physics and integral equations".
- Carl Bender and Steven Orszag,
  "Advanced mathematical methods for scientists and engineers".

- Ray Wylie and Louis Barrett,
  "Advanced engineering mathematics".
- Dennis Zill and Michael Cullen,
  "Differential equations with boundary value problems".
- Kent Nagle and Edward Saff,
  "Fundamentals of differential equations".
- Yehuda Pinchover and Jacob Rubinstein,
  "An introduction to partial differential equations".

Introduction:

- S. L. Sobelov,
  "Partial differential equations of mathematical physics".
- E. C. Zachmanoglou and Dale Thoe,
  "Introduction to partial differential equations with applications".
- K. F. Riley, M. P. Hobson, and S. J. Bence,
  "Mathematical methods for physics and engineering".
- Walter Strauss,
  "Partial differential equations: An introduction".
- Robert Borrelli and Courtney Coleman,
  "Differential equations: A modelling perspective".

Introduction:

- Polyanin's "handbook" series:
  - Andrei Polyanin and Valentin Zaitsev,
    "Exact solutions for ordinary differential equations".
  - Andrei Polyanin and Valentin Zaitsev,
    "Nonlinear partial differential equations".
  - Andrei Polyanin,
    "Linear partial differential equations for scientists and engineers".
  - Andrei Polyanin, Valentin Zaitsev, and A. Moussiaux,
    "First order partial differential equations".
- In addition, Google can quite easily direct you to lots of online notes on PDEs — almost all of very high quality.

Introduction:

This is also a topic on which Wikipedia is reasonably trustworthy.

See for instance:

- https://en.wikipedia.org/wiki/Partial_differential_equation
- https://en.wikipedia.org/wiki/First_order_partial_differential_equation
- https://en.wikipedia.org/wiki/Separable_partial_differential_equation
- https://en.wikipedia.org/wiki/Separation_of_variables
- https://en.wikipedia.org/wiki/Method_of_characteristics
- https://en.wikipedia.org/wiki/Fourier_series
- https://en.wikipedia.org/wiki/Convergence_of_Fourier_series

Basic definitions

Basic definitions:

Definition

PDE:

A partial differential equation (PDE) is an equation involving one or more unknown functions, (the "fields"), of two or more independent variables, ("position" and "time", or "position"), and derivatives of the unknown functions with respect to the independent variables.

Dependent and independent variables:

- We shall generally consider there to be two independent variables, denoted either by x and y, or by t and x, (sometimes [rarely] by x_1 and x_2), or (more commonly) by x^1 and x^2.
- In differential geometry, applied mathematics, and theoretical physics it is typically most common to use superscripts to denote different independent variables, x^1 and x^2.
- A potential problem with this convention is that you then you have to be careful to not get confused with exponents;

$$\text{That is:} \qquad x^2 \neq (x)^2 \;!$$

Dependent and independent variables:

- Nevertheless, the superscript convention is so well established, [in both applied mathematics and theoretical physics], that I will consistently adopt it throughout these notes.
- So get used to seeing things like x^1 and x^2.
- Notation such as x_1 and x_2 is to be discouraged. (You might sometimes still see such notation — just don't copy it.)
- The generalization of results and methods to more than two independent variables will be "straightforward" and is left to you.

Dependent and independent variables:

- (Actually "straightforward" is a "code word" that you should learn to recognize — it means that extensions to more than two dimensions are in principle easy — but in practice can turn quickly into computational nightmares.)
- This means that almost everything we will be doing is either in $(1+1)$ dimensions [one space dimension, plus one time dimension] or in two space dimensions — $(2+0)$ dimensions if you want to be difficult.

Dependent and independent variables:

- Some constructions and techniques do depend specifically on the number of dimensions — watch out; I'll try to give you appropriate warnings.
- There are some features of $(3+1)$ dimensions,
  [three space dimensions, plus one time dimension],
  the universe we live in, that are just not adequately captured by the $(1+1)$ dimensional simplification.

Dependent and independent variables:

- But in cases of high symmetry one can often reduce the effective number of dimensions of the problem:
  - Spherical symmetry in (3+1) dimensions
    $\Longrightarrow$ everything depends on at most distance from the centre, and possibly on time,
    $\Longrightarrow$ effectively (1+1) dimensions.
  - Axial symmetry in (3+1) dimensions
    $\Longrightarrow$ everything depends on at most distance from the axis, and possibly on time,
    $\Longrightarrow$ effectively (1+1) dimensions.
  - Planar symmetry in (3+1) dimensions
    $\Longrightarrow$ everything depends on at most x and y but not on z, and possibly on time,
    $\Longrightarrow$ effectively (2+1) dimensions.

Dependent and independent variables:

- We shall also, [for many of these lectures, excluding the section on Frobenius systems], assume there is only one dependent variable.
- (In physics language, we are dealing with only one "field" such as pressure, or density, or displacement.)
- We shall use any of the symbols U, u, V, v ... to denote that variable.
- The generalization of results and methods to more than one dependent variable will be "straightforward" and is left to you.
- (Notice that code word again. Be very afraid.)

Dependent and independent variables:

- Physically, generalizing to more than one dependent variable would be useful in situations such as:
  - electric and magnetic fields [the Maxwell equations],
  - in Einstein's theory of gravity [the general relativity, where there are 10 inter-connected gravitational "potentials"],
  - or in fluid mechanics [where, at a minimum, you have to keep track of both density and velocity].
- Still, one step at a time, in this course we will mostly stick to one dependent variable.

Dependent and independent variables:

- Warning: you will soon see that the mathematical theory of (general) PDEs is much less well-developed than the mathematical theory of general ODEs.
- When it comes to PDEs, the mathematical situation is still pretty much that we have a lot of information about a large number of individual special cases — and relatively little information about truly general situations.

Partial derivatives 1:

Notation

There are many different notations used for partial derivatives.
Variously used, but completely equivalent, notations are:

$$
\begin{aligned}
D_{(0,1)} U &= D_1 U = D_x U = \frac{\partial U}{\partial x} = \partial_x U = U_x = \frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x^1}. \\
D_{(1,0)} U &= D_2 U = D_y U = \frac{\partial U}{\partial y} = \partial_y U = U_y = \frac{\partial U}{\partial x_2} = \frac{\partial U}{\partial x^2}. \\
D_{(1,1)} U &= D_1 D_2 U = D_x D_y U = \frac{\partial^2 U}{\partial x\, \partial y} = \partial_x \partial_y U = U_{xy} \\
&= \frac{\partial^2 U}{\partial x_1\, \partial x_2} = \frac{\partial^2 U}{\partial x^1\, \partial x^2}. \\
\ldots \quad & \ldots\ldots\ldots \quad \ldots\ldots\ldots
\end{aligned}
$$

Partial derivatives 2:

Notation

There are many different notations used for partial derivatives.
Variously used, but completely equivalent, notations are:

$$
\begin{aligned}
D_{(2,1)} U &= D_1^2 D_2 U = D_x^2 D_y U = \frac{\partial^3 U}{(\partial x)^2 \; \partial y} = \partial_x^2 \partial_y U = U_{xxy} \\
&= \frac{\partial^3 U}{(\partial x_1)^2 \; \partial x_2} = \frac{\partial^3 U}{(\partial x^1)^2 \; \partial x^2}. \\
\dots \quad & \quad \dots\dots\dots \quad \dots\dots\dots
\end{aligned}
$$

and so on.

Learn to recognize all these variant notations.

Partial derivatives 3:

Notation

I will try to standardize notation in this course to be as follows:

$$
\begin{aligned}
U_x &= \partial_x U = \frac{\partial U}{\partial x}. \\
U_y &= \partial_y U = \frac{\partial U}{\partial y}. \\
U_{xy} &= \partial_x \partial_y U = \frac{\partial^2 U}{\partial x\, \partial y}. \\
U_{xxy} &= \partial_x^2 \partial_y U = \frac{\partial^3 U}{(\partial x)^2\, \partial y}. \\
\dots &\quad \dots\dots\dots\dots\dots\dots
\end{aligned}
$$

Partial derivatives 4:

Notation

I will also sometimes use:

$$U_{,i} = \partial_i U = \frac{\partial U}{\partial x^i},$$

especially when I have more than two independent variables to deal with.

These "standard" notations are the most common of the notations you are likely to run into when reading books or scientific articles.

Notes:

1. So long as the function $U(x,y)$ is C^s (meaning that all partial derivatives up to order s exist and are continuous), then the seqience in which you take the partial derivatives in an r-th order derivative, for any $r \leq s$, does not matter.
2. In the usual spirit of applied mathematics and theoretical physics, we shall take all our functions to be smooth enough, in the sense that all partial derivatives that we may happen to need will be assumed to exist and to be continuous.
3. That is, for all practical purposes:

$$\boxed{\frac{\partial^2 U}{\partial x\, \partial y} = \frac{\partial^2 U}{\partial y\, \partial x}.}$$

Definition

Order of a PDE:
The order of a PDE is the highest order of differentiation appearing in the PDE.
Do not confuse this with the degree of terms appearing in the equation.

Order:

- If we wish to refer to a general derivative of U of the m-th order, without regard to the precise variables that are being used in the differentiation, we shall write $U^{(m)}$.
- That is, $U^{(m)}$ stands generically for an m-th order derivative, and we can write

  $$F\left(x, y, U, U^{(1)}, U^{(2)}, ..., U^{(n)}\right) = 0$$

  as the general form of an n-th order PDE,
  with one dependent variable U,
  and two independent variables x and y.
- Note that $U^{(2)}$ for instance could mean any (or all) of U_{xx}, U_{xy}, U_{yy}.

Order:

Definition

***n*-th order PDE:**
An n-th order PDE is a relation of the form

$$F\left(x, y, U, U^{(1)}, ..., U^{(n)}\right) = 0.$$

Order is a statement about how many times you will need to differentiate the dependent variable to even write down the PDE.

Linearity:

Definition

Linear PDE:
An n-th order PDE,

$$F\left(x, y, U, U^{(1)}, ..., U^{(n)}\right) = 0,$$

is n-th order linear iff (if and only if) it is of the form:

$$a_n(x, y)\ U^{(n)} + a_{n-1}(x, y)\ U^{(n-1)} + ... + a_0(x, y)\ U + b(x, y) = 0,$$

with $a_n(x, y)$ not identically zero.

It is homogeneous iff (if and only if) $b(x, y) = 0$.

Linearity:

Linearity is a statement about the manner in which the dependent field $U(x, y)$ appears in the PDE.

Simple examples:

- The Klein–Gordon equation:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = m^2 U,$$

  is a second-order linear homogeneous equation.

- The Sine–Gordon equation:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = \sin U,$$

  is a second-order non-linear equation.

Linearity:

- The Korteweg–deVries (KdV) equation:

$$\frac{\partial^3 U}{\partial x^3} + 6U \, \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0.$$

  This equation arises as one particular model for describing shallow water waves.
  It is a third-order non-linear PDE.

- Both KdV and SG have become very prominent as model equations for analyzing problems involving solitary waves (solitons).

Linearity:

- The wave equation:

$$\frac{\partial^2 U}{\partial t^2} - c^2 \, \frac{\partial^2 U}{\partial x^2} = 0,$$

  is a second-order linear and homogeneous equation which is important in the description of many travelling wave phenomena.

- Laplace's equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$

  is a second-order linear and homogeneous equation which is important in the description of many electrostatic and gravitational phenomena.

- The diffusion equation (heat equation):

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = 0,$$

  is an important second-order linear and homogeneous equation which describes many transfer problems, such as diffusion (gaseous or chemical) or heat transfer.

  - The Boltzmann equation of statistical mechanics is an equation of this type.
  - The diffusion equation is also important in various "random walk" models ("drunkard's walk"), and underlies important financial mathematics in the theory of financial derivative pricing — the Black–Scholes differential equation is of this type.
  - Similarly "genetic drift" in population dynamics is governed by diffusion-type equations.

Linearity:

- The (free) Schroedinger equation:

$$\frac{\partial^2 U}{\partial x^2} - i\frac{\partial U}{\partial t} = 0,$$

  is a complexified version of the diffusion equation.
  It is again second-order, and linear, and homogeneous.
  This equation underlies all of quantum physics, and a good hunk of modern technology (in particular, all solid state electronics).
- The Maxwell equations of classical electromagnetism are first-order linear inhomogeneous PDEs in 6 dependent variables (the electric and magnetic fields) and 4 independent variables (space + time).
- Linear PDEs are extremely useful, and quite a lot is known about them.

Quasi-linearity:

Definition

Quasi-linear PDE:

An n-th order PDE,

$$F\left(x, y, U, U^{(1)}, ..., U^{(n)}\right) = 0,$$

is quasi-linear if it is linear in the n-th order derivatives.

(It is allowed to be nonlinear in lower-order derivatives, and even the coefficients of the n-th order derivatives are allowed to depend on the lower-order derivatives in a nonlinear manner).

Quasi-linearity:

Definition

Quasi-linear PDE:

That is, letting $U_A^{(n)}$ denote the various possible n-th order derivatives, a quasi-linear PDE is described by an equation of the form

$$\sum_A C^A\left(x, y, U, U^{(1)}, ..., U^{(n-1)}\right) \; U_A^{(n)}$$

$$+\tilde{F}\left(x, y, U, U^{(1)}, ..., U^{(n-1)}\right) = 0,$$

with the $C^A(x, y, U, U^{(1)}, ..., U^{(n-1)})$ not all identically zero.

Quasi-linearity:

Examples:

- First-order quasi-linear PDEs are of the form

$$\alpha(x, y, U)\, \partial_x U + \beta(x, y, U)\, \partial_y U + \gamma(x, y, U) = 0.$$

  Quite a lot is known about solving PDEs of this type.
  (There is a technique called the method of characteristics, which we will at best only mention in Part 1.)
  Specific examples:
  - $x u_x + (x + y) u_y = u + 1.$
  - $x u_x + u^4 u_y = u^3.$

Quasi-linearity:

- Second-order quasi-linear PDEs are of the form

$$a(x,y,U,U_x,U_y)\,U_{xx} + b(x,y,U,U_x,U_y)\,U_{xy} + c(x,y,U,U_x,U_y)\,U_{yy}$$
$$+d(x,y,U,U_x,U_y) = 0.$$

  We shall see these PDEs again later on in the course, and under a different name.

- The sine–Gordon equation:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = \sin U,$$

  is a specific second-order quasi-linear equation.

Quasi-linearity:

- The quasi-linear Klein–Gordon equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + m^2 \; U = \lambda \; U^3,$$

is another commonly occurring second-order quasi-linear equation.

(**Challenge:** Find the exact plane wave solutions for this PDE, and no, I am not talking about sine and cosine anymore...)

(**Challenge:** What is the connection between this PDE and the Higgs particle of elementary particle physics?)

Quasi-linearity:

- Mathematicians now often talk about f–Gordon equations where

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = f(x, t, U, U_x, U_y).$$

Here $f(x, t, U, U_x, U_y)$ is an arbitrary nonlinear function of its arguments.

Quasi-linearity:

- The Korteweg–deVries (KdV) equation

$$\frac{\partial^3 U}{\partial x^3} + 6U\,\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0,$$

  is quasi-linear because the U_{xxx} term occurs linearly.

Quasi-linearity:

- Keeping the highest-order derivatives linear is sometimes enough to let us prove useful theorems.
- Quite a lot (comparatively speaking) is known about quasi-linear PDEs.

Quasi-linearity:

- The Einstein equations of classical general relativity are second-order quasi-linear PDEs in 10 dependent variables
  (the "metric" describing the spacetime geometry)
  and 4 independent variables (space + time).

Boundary conditions and/or Initial conditions:

- Many PDEs arise in problems for which, in addition to defining the PDE to solve, there are naturally occurring conditions, called "boundary conditions" (BCs), [or sometimes "initial conditions" (ICs)], that the solution must also satisfy.

- In many common specific cases, the PDEs and their associated BCs and/or ICs can be classified into standard types (with names such as "elliptic", "hyperbolic", "parabolic") for which the whole problem, PDE and associated BCs and/or ICs, can be shown to have a unique solution.

Boundary conditions/Initial conditions:

The distinction between boundary conditions and initial conditions makes sense only if you have a problem involving both space and time.

- **Initial conditions** provide constraints on the dependent variables at some initial instant in time, throughout some region of space.
- **Boundary conditions** provide constraints on the dependent variables at some place in space, throughout some interval of time.
- **Radiation conditions** provide constraints on the dependent variables in terms of incoming [or outgoing] wave motion.

Boundary conditions/Initial conditions:

If you are into special or general relativity — Initial conditions are specified on spacelike surfaces, boundary conditions are specified on timelike surfaces, and radiation conditions are specified on lightlike surfaces [null surfaces].

And to add confusion, sometimes the phrase “boundary conditions” is used indiscriminately to refer to all three types.

Boundary conditions/Initial conditions:

Suppose now we denote the boundary by the curve $(x(s), y(s))$, or more generally the surface $\vec{x}(\sigma)$, and denote the normal derivative to the boundary by ∂_n. Standard terminology is:

- **Normal derivative:**
$$\partial_n = \hat{\mathbf{n}} \cdot \nabla$$

We then have:

- **Dirichlet BC:** The value of the dependent variable is specified on the boundary:
$$U(\vec{x}(\sigma)) = f(\sigma).$$
- **Neumann BC:** The value of the normal derivative of the dependent variable is specified on the boundary:
$$\partial_n U(\vec{x}(\sigma)) = f(\sigma).$$
- **Robin BC:** Some linear combination of these...

- **Robin BC:** Some linear combination of the dependent variable and its normal derivative is specified on the boundary:

$$a(\sigma)\; U(\vec{x}(\sigma)) + b(\sigma)\; \partial_n U(\vec{x}(\sigma)) = f(\sigma).$$

There is a vast literature on solving equations of these types.
Look, for example, in:

- Courant, R. and D. Hilbert,
  Methods of Mathematical Physics, Vols 1 and 2.

I'll have a lot more to say about these issues soon.

Summary (Order, linearity, etc...):

Reminder

- *The order of a PDE is the order of the highest derivative appearing in the equation.*
- *PDE is linear if it is of the first degree in the dependent variables and their derivatives.*
- *A linear PDE is homogeneous if every term in its expression is linear in the dependent variables and their derivatives.*
- *A PDE is quasi-linear if the highest-order "derivative part" is linear, though the coefficients and the subleading terms are allowed to be nonlinear.*
- *If a PDE is nonlinear the question of whether or not it is homogeneous is best regarded as meaningless.*

Exercises

Exercises (On order, linearity, etc...):

- Classify the following PDEs by stating their order, and whether they are linear or not.
- If they are linear, classify then as to whether they are homogeneous or not.
- If they are nonlinear, classify then as to whether they are quasi-linear or not.

a. $V^2\ V_{xy} + V_x\ V_y + (x^2 - y^2)V = 3xy.$

b. $U_{xxz} - 2(x+z)U_{xyz} - U_{xx} + \sin(xyz)U_{xx} = \cos(U)$

c. $U_t - UU_{xx} + 12xU_x = U.$

d. $Y_{xxx} - \cos Y = Y_t.$

e. $V_{xt} - \sin V = \exp(x+t).$

Exercises (On order, linearity, etc...):

f. $Y_{xx} + \cos(xy) Y_{yxy} = Y + \ln(x^2 + y^3)$.

g. $U_t = U_{xx} - 12U\ U_x$.

h. $V_{yx} + V_x + V_y = V_{xyy}$.

i. $U_{tt} - \cos(U_x) = U$.

j. $\cos x \cdot U_x + \sin t \cdot U_t = U$.

k. Schrodinger equation (with potential):

$$-i\partial_t \psi = \frac{1}{2m} \nabla^2 \psi + V(x)\psi.$$

Exercises (On order, linearity, etc...):

l. Monge–Ampere equation (2 variable):

$$u_{xx} u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y).$$

m. Monge–Ampere equation (multi-variable):

$$\det \left[\frac{\partial^2 u}{\partial x^i \, \partial x^j} \right] = f\left(x^i, u, \frac{\partial u}{\partial x^i} \right).$$

n. Navier–Stokes equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}.$$

o. Tricomi equation:

$$U_{xx} + y \; U_{yy} = 0.$$

Exercises (On order, linearity, etc...):

p. Frobenius–Mayer equation (special case, one dependent variable):

$$\frac{\partial U}{\partial x^i} = F_i(x, U).$$

q. Biharmonic equation:

$$\nabla^4 \Psi = 0.$$

That is, $(\nabla^2)^2\ \Psi = 0$, or more explicitly:

$$\left[\partial_x^2 + \partial_y^2 + \partial_z^2\right]^2 \Psi = 0.$$

r. Benjamin–Bona–Mahony equation:

$$u_t + u_x + u u_x - u_{xxt} = 0.$$

Exercises (On order, linearity, etc...):

s. Chaplygin equation:

$$u_{xx} + \frac{c^2\ y^2}{c^2 - y^2}\ u_{yy} + y\ u_y = 0.$$

t. Boussinesq equation:

$$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}.$$

u. Euler–Darboux equation:

$$u_{xy} + \frac{\alpha\ u_x - \beta\ u_y}{x - y} = 0.$$

v. Korteweg–deVries–Burger:

$$u_t + 2uu_x - \nu\ u_{xx} + \mu\ u_{xxx} = 0.$$

Exercises (On order, linearity, etc...):

w. Kirchever–Novikov equation:

$$\frac{u_t}{u_x} = \frac{1}{4}\frac{u_{xxx}}{u_x} - \frac{3}{8}\frac{u_{xx}^2}{u_x^2} + \frac{3}{8}\frac{4u^3 - g_2 u - g_3}{u_x^2}.$$

(Start by simplifying this a little.)

x. Lin–Tsien equation:

$$2u_{tx} + u_x\, u_{xx} - u_{yy} = 0.$$

Exercises (On order, linearity, etc...):

y. Monge–Ampere equation (generalized):

$$
\begin{aligned}
&E(x, y, U, U_x, U_y)\left[U_{xx}U_{yy} - U_{xy}^2\right] \\
&+A(x, y, U, U_x, U_y)\; U_{xx} + B(x, y, U, U_x, U_y)\; U_{xy} \\
&\qquad +C(x, y, U, U_x, U_y)\; U_{yy} \\
&\qquad\qquad +D(x, y, U, U_x, U_y) = 0
\end{aligned}
$$

Even more generally (multi variable case):

$$
E(x^i, U, \partial_i U)\,\det\left[\frac{\partial^2 U}{\partial x^i\;\partial x^j}\right] + \sum_{ij} A^{ij}(x^i, U, \partial_i U)\; U_{,ij}
$$

$$
+D(x^i, U, \partial_i U) = 0.
$$

z. Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Iterate these Cauchy–Riemann equations to find a pair of PDEs that decouple — they depend only on u, and only on v, but not both.

End: