Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs — Autumn 2024

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Outline:

1. Administrivia

2. General solutions
   - Definitions
   - Examples of general solutions
   - Exercises regarding general solutions

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

General solutions

General solution:

Unlike ODEs, the notion of a general solution of a PDE can get very complicated, very quickly.
In these lectures, when the term "general solution" is used,
it will be meant in the following special sense:

Definition (General solution)

A solution $U(x, y)$ of an n-th order PDE with a single dependent variable

$$F\left(x, y, U(x, y), U^{(1)}, U^{(2)}, ., ., ., U^{(n)}\right) = 0$$

is a "general solution" if U depends on n arbitrary independent functions.

General solution:

Warning

Note "independent functions" not "independent constants".

This is a direct extension of the notion of a general solution taken from the case of ODEs:

- Recall that for an ODE, a general solution is a solution depending on n independent constants: and recall that we arrived at this idea by noting that, in principle, to solve an n-th order DE, we essentially need to integrate n times — and each integration introduces an arbitrary constant.
- The same applies of course to a PDE — to solve it, we in principle must integrate n times, and each integration introduces a function (rather than a constant). The examples below illustrate this fact.

General solution:

- When it comes to a general PDE, or general systems of PDEs, the precise situation regarding a general solution can only be clearly stated using the relatively sophisticated work of Riquier and Janet (brief comments in the next chapter).
  It is not appropriate to describe this in MATH 301.

Reminder

Even for ODEs, in the nonlinear case life is a lot more complicated than you might at first suspect.

Examples of general solutions:

Here are some simple examples of "general solutions":

Example 1:
The equation

$$\frac{\partial U}{\partial x} = 0,$$

Keep in mind what the partial derivative means — you are differentiating U with respect to x, treating y as if it were constant.
The general solution is this:

$$U(x, y) = G(y).$$

A different "constant" for each independent value of y...

Examples of general solutions:

Example 2:
The equation

$$\frac{\partial U}{\partial x} = f(x, y), \qquad \text{for some given } f(x, y).$$

Keep in mind what the partial derivative means — you are differentiating U with respect to x, treating y as if it were constant.

Examples of general solutions:

To regain U, then it would seem that we should integrate with respect to x, again keeping y constant:

$$U(x, y) = \int_{y \text{ constant}} f(x, y) \, \mathrm{d}x + G(y).$$

Here G is an arbitrary "constant", which, since y is considered constant, is allowed to be a different arbitrary "constant" for each specific value of y.

That is, $G(y)$ is generally a function of y.

Examples of general solutions:

Introducing the dummy variable $\bar{x}$ we can make this general solution more explicit as:

$$U(x, y) = \int_{x_0}^{x} f(\bar{x}, y)\, \mathrm{d}\bar{x} + G(y).$$

This is the general solution to

$$\frac{\partial U}{\partial x} = f(x, y), \qquad \text{for some given } f(x, y).$$

Example 3:
The equation

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = g(x, y),$$

where $g(x, y)$ is a given function.
Here it will pay to change the independent variables, to new independent variables s, t defined by

$$s = x + y;$$

$$t = x - y.$$

So that

$$x = \frac{s+t}{2}; \qquad y = \frac{s-t}{2}.$$

Examples of general solutions:

But by the [2-variable] chain rule

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial}{\partial t}.$$

Then it is easy to show that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t};$$

and similarly

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s} - \frac{\partial}{\partial t};$$

or equivalently

$$\frac{\partial}{\partial s} = \frac{1}{2}\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\};$$

$$\frac{\partial}{\partial t} = \frac{1}{2}\left\{\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right\}.$$

Examples of general solutions:

Hence the original PDE is now converted to

$$\frac{\partial U}{\partial s} = \frac{1}{2}\, g\left(\frac{s+t}{2}, \frac{s-t}{2}\right) = G(s,t)$$

which can now be solved in general as in the first example.
Doing so yields

$$U(s,t) = \int_{t \text{ constant}} G(s,t)\, \mathrm{d}s + H(t)$$

which we first re-write (explicitly using the dummy variable $\bar{s}$) as

$$U(s,t) = \frac{1}{2} \int_{s_0}^{s} g\left(\frac{\bar{s}+t}{2}, \frac{\bar{s}-t}{2}\right)\, \mathrm{d}\bar{s} + H(t).$$

Examples of general solutions:

Now follow this by a change of independent variables back to x and y to produce our final answer:

$$U(x,y) = \frac{1}{2} \int_{s_0}^{x+y} g\left(\frac{\bar{s} + [x-y]}{2}, \frac{\bar{s} - [x-y]}{2} \right) \, \mathrm{d}\bar{s} + H(x-y).$$

Remember the PDE we are solving is:

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = g(x,y),$$

where $g(x,y)$ is a given function.

Examples of general solutions:

Example 4:
The equation

$$\frac{\partial^2 U}{\partial x\,\partial y} = H(x, y)$$

for a given function H.
Take the LHS to be

$$\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\right]$$

and proceed as in the first example, integrating with respect to x, treating y as constant:

$$\frac{\partial U}{\partial y} = \int_{y \text{ constant}} H(x, y)\,\mathrm{d}x + g(y),$$

where g is an arbitrary function.

Examples of general solutions:

Now integrate with respect to y, treating x as a constant:

$$U(x,y) = \int_{x \text{ constant}} \left[\int_{y \text{ constant}} H(x,y)\,\mathrm{d}x \right] \mathrm{d}y + G(y) + F(x)$$

where F is another arbitrary function, and G is the integral of$g$ (and so is an arbitrary function).

Examples of general solutions:

In terms of dummy variables $\bar{x}$ and $\bar{y}$ our general solution can be rewritten in the explicit form:

$$U(x, y) = \int_{y_0}^{y} \left[\int_{x_0}^{x} H(\bar{x}, \bar{y}) \, \mathrm{d}\bar{x} \right] \mathrm{d}\bar{y} + G(y) + F(x).$$

This is the general solution to

$$\frac{\partial^2 U}{\partial x \, \partial y} = H(x, y),$$

for a given function H.
Exercise: Check the special case $H(x, y) = 0$.

Examples of general solutions:

Example 5:
The equation

$$\frac{\partial^2 U}{\partial x^2} = H(x, y),$$

for a given function $H(x, y)$.
Proceeding as before, write

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)$$

so

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = H(x, y).$$

Examples of general solutions:

Integrating [twice] with respect to x and keeping y fixed, we find

$$U(x,y) = \int_{y \text{ constant}} \left[\int_{y \text{ constant}} H(x,y) \, \mathrm{d}x \right] \mathrm{d}x + x\, G(y) + F(y),$$

where G and F are arbitrary “constants”.

Exercise: Check the special case $H(x,y) = 0$.

Examples of general solutions:

In terms of dummy variables $\bar{x}$ and $\tilde{x}$ our general solution can be rewritten in the explicit form:

$$U(x, y) = \int_{\tilde{x}}^{x} \left[\int_{x_0}^{\tilde{x}} H(\bar{x}, y) \, \mathrm{d}\bar{x} \right] \mathrm{d}\tilde{x} + x \; G(y) + F(y).$$

This is the general solution to

$$\frac{\partial^2 U}{\partial x^2} = H(x, y),$$

for a given function $H(x, y)$.

From these five examples the general pattern should be obvious.

Examples of general solutions:

Comment

Consider the general change of independent variables (that is, the general two-dimensional change of coordinates):

$$(x, y) \to (u, v) = (u(x, y), v(x, y))$$

What happens to the partial derivatives?
The general rule is this

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\,\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\,\frac{\partial}{\partial v};$$

and

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\,\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\,\frac{\partial}{\partial v}.$$

This is (should be) obvious — think of it as an application of the chain rule (multi-variable chain rule).

Examples of general solutions:

Comment (continued)

Similarly if we consider the inverse transformation

$$(u, v) \to (x, y) = (x(u, v), y(u, v))$$

we see

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\,\frac{\partial}{\partial y};$$

and

$$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\,\frac{\partial}{\partial y}.$$

Examples of general solutions:

Comment

You should also be prepared for notation such as

$$(x, y) \to (u, v) = (u(x, y), v(x, y))$$

$$\partial_x = (\partial_x u)\, \partial_u + (\partial_x v)\, \partial_v;$$

$$\partial_y = (\partial_y u)\, \partial_u + (\partial_y v)\, \partial_v.$$

and

$$(u, v) \to (x, y) = (x(u, v), y(u, v))$$

$$\partial_u = (\partial_u x)\, \partial_x + (\partial_u y)\, \partial_y;$$

$$\partial_v = (\partial_v x)\, \partial_x + (\partial_v y)\, \partial_y.$$

Exercises regarding general solutions:

Reminder

- *The general solution to an ODE of the n-th order contains n arbitrary and independent constants.*
- *For PDEs the situation is much more complicated, but nevertheless we will define a general solution of a single PDE of the n-th order in a single unknown U as a solution involving n arbitrary functions.*
- *This of course is not the best definition, but it will do here.*

Exercises regarding general solutions:

Reminder

- *In the case of an ODE the general solution completely defines its corresponding ODE in the sense that, given a function depending on n independent and arbitrary constants, there should only be one n-th order ODE which has that function as its general solution.*

- *[To see this, recall that we considered an ODE as a means of encoding all the derivatives of its solution, the n arbitrary constants being the first few derivatives, at $x = 0$ say, that are not defined by the ODE].*

Exercises regarding general solutions:

Reminder

- *In a similar fashion, given a function $u(x, y)$ which also involves n independent functions, there will be a (hopefully unique) PDE of n-th order that will have that function as its general solution.*

- *One of the questions below asks you to find the corresponding PDE for given general solutions.*

From general solution to PDE

Consider the general solution

$$u = f(x - y).$$

Then

$$\frac{\partial u}{\partial x} = f'(x - y); \qquad \frac{\partial u}{\partial y} = -f'(x - y).$$

Eliminate f', obtaining

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

This PDE now makes no reference to f, and the general solution of this PDE is the equation you started from.

Exercises regarding general solutions:

- Using this technique, eliminate the arbitrary functions from the following general solutions, and so obtain partial differential equations of which they are the general solution:

  a. $u = f(x+y)$.
  b. $u = g(xy)$.
  c. $u = f(x+y) + g(x-y)$.
  d. $u = x^n\, h(y/x)$.
  e. $v = g(x^2+y^2)$.
  f. $v = f(x^2-y^2)$.
  h. $v = f(x^2-y^2) + g(x^2+y^2)$.
  i. $v = h(2x-y) - g(2x+y)$.

Exercises regarding general solutions:

- Now consider a general solution specified by the system of two equations:

$$u(x,y) = \alpha(x,y)\, x + w(\alpha(x,y))\, y + v(\alpha(x,y));$$

$$0 = x + w'(\alpha(x,y))\, y + v'(\alpha(x,y)).$$

Eliminate the arbitrary functions $w(\alpha)$ and $v(\alpha)$, and the parameter α itself, to obtain a PDE for $u(x,y)$.

(You should find a particularly simple example of a Monge–Ampere equation.)

See Courant and Hilbert — Volume 2 page 10.

Exercises regarding general solutions:

Solution to previous exercise:

- We start with:

$$u(x,y) = \alpha(x,y)\,x + w(\alpha(x,y))\,y + v(\alpha(x,y));$$
$$0 = x + w'(\alpha(x,y))\,y + v'(\alpha(x,y)).$$

- So:

$$u_x = \alpha + [x + w'(\alpha)y + v'(\alpha)]\alpha_x = \alpha;$$
$$u_y = w(\alpha) + [x + w'(\alpha)y + v'(\alpha)]\alpha_y = w(\alpha);$$
$$u_y = w(u_x).$$

- Then:

$$u_{xy} = w'(u_x)u_{xx}; \qquad u_{yy} = w'(u_x)u_{xy};$$
$$\frac{u_{xx}}{u_{xy}} = \frac{u_{xy}}{u_{yy}}$$

- Finally:

$$u_{xx}u_{yy} - u_{xy}^2 = 0$$

Exercises regarding general solutions:

Exercise:

- Suppose you are given a class of functions

$$y(x : \vec{a}) = f(x : a_1, a_2, \dots, a_n)$$

  of the single variable x, where the class of functions is parameterized by arbitrary parameters a_1, a_2, ..., a_n, denoted collectively by $\vec{a}$.
- Suppose further that the parameters come under the heading of "arbitrary and independent", namely, suppose that the following determinant is non-zero $(i, j = 1, \dots, n)$:

$$\det \left[\left(\frac{\partial}{\partial x} \right)^i \frac{\partial}{\partial a_j} f(x : \vec{a}) \right] \neq 0. \qquad (C)$$

- Then you can easily prove that $y(x : \vec{a})$ must be the general solution of some ODE of the n-th order.

Exercises regarding general solutions:

- You do this effectively by eliminating the constants a_k, $k = 1, 2, \ldots, n$.
- Consider the n equations:

$$
\begin{array}{rcl}
y & = & y(x : a_1, a_2, .., a_n) \\
y' & = & f'(x : a_1, a_2, .., a_n) \\
y'' & = & f''(x : a_1, a_2, .., a_n) \\
. & . & ... \\
. & . & ... \\
. & . & ... \\
y^{(n-1)} & = & f^{(n-1)}(x : a_1, a_2, .., a_n)
\end{array}
$$

- These are n equations relating the n variables $y, y', y'', .., y^{(n-1)}$ to the n "variables" $a_1, a_2, .., a_n$.

- Because of the condition (C) above, the inverse function theorem guarantees that you can (at least locally) solve these equations to find the variables $a_1, a_2, .., a_n$ as functions of the variables $y, y', y'', .., y^{(n-1)}$ and x.
- That is:

$$a_k = A_k(x : y, y', y'', ...y^{(n-1)})$$

for $k = 1, 2, ..., n$ and some functions A_k of the indicated variables..

Exercises regarding general solutions:

- Now use these functions to eliminate the variables $a_1, a_2, .., a_n$ in the expression for the n-th derivative of y:

$$y^{(n)} = f^{(n)}(x : a_1, a_2, .., a_n)$$

  in favour of the derivatives $y, y', y'', .., y^{(n-1)}$.

- That is

$$y^{(n)} = f^{(n)}\left(x : A_i(x : y, y', y'', ...y^{(n-1)})\right).$$

- Doing so, you will end up with a relation between the derivatives of the function y of the form:

$$y^{(n)} = G(x, y', y'', ..., y^{(n-1)}),$$

  which is an ODE in y of order n.

- (In fact it's even guaranteed to be quasi-linear).

Exercises regarding general solutions:

Challenge:

Can you now set up an analogous way of obtaining a PDE?

- Specifically, consider the general class of functions

$$u = f(x, y; \alpha, \beta)$$

- By differentiating with respect to x and y, and then appealing to the inverse function theorem, argue that this general class of functions is the solution set of the generic first-order PDE

$$F(x, y, u, u_x, u_y) = 0.$$

Exercises regarding general solutions:

- What happens for the three-parameter general class of functions

$$u = f(x, y; \alpha, \beta, \gamma)?$$

- Develop a general formalism for going from a parameterized class of "solutions" to the PDE that "generates" that solution class.

- (When all else fails, look up Courant and Hilbert, volume 2, pp. 8 ff. for some hints...)

Exercises regarding general solutions:

From PDE to general solution

By integrating out the partial derivatives in the following PDEs, find the general solution.

a. $U_{xy} = y\ U_x^3$.

b. $U_{xy} = xy\ U_y$.

c. $U_{xy} = y\ U_y + x^3 y^2$.

d. $U_{xx} = y\ U_x + xy$.

e. $U_x = U_y$.

f. $\alpha\ U_x + \beta\ U_y = 0$. (Treat α and β as given constants.)

g. $U_x\ g_y(x,y) - U_y\ g_x(x,y) = 0$. (Treat $g(x,y)$ as given.)

h. $U_{xxyy} = 0$.

This exercise illustrates the rather complex way that the arbitrary functions could appear in the general solution.

Exercises regarding general solutions:

Now try to find the general solution for:

i. $\alpha(U)\, U_x - \beta(U)\, U_y = 0$.

Solution: Look at lines of constant U, solving $U(x, y) = k$, then

$$\frac{dy}{dx} = \frac{(\partial y/\partial U)}{(\partial x/\partial U)} = \frac{(1/U_y)}{(1/U_x)} = \frac{U_x}{U_y} = \frac{\beta(U)}{\alpha(U)}$$

So the lines of constant U are straight lines and in fact

$$\frac{y - y_0}{x - x_0} = \frac{\beta(U)}{\alpha(U)}$$

which we can rearrange to

$$(y - y_0)\alpha(U) = (x - x_0)\beta(U)$$

That is

$$x\,\beta(U) - y\,\alpha(U) = F(U).$$

You will just have to be satisfied with an implicit relation for $U(x, y)$ in terms of some arbitrary function $F(U)$.

Exercises regarding general solutions:

Now try to find the general solution for:

j. $U_x \frac{\mathrm{d}g}{\mathrm{d}y}(x,y,U) - U_y \frac{\mathrm{d}g}{\mathrm{d}x}(x,y,U) = 0.$ [Treat $g(x,y,U)$ as given.]

Solution: Note the PDE is equivalent to

$$(U_x, U_y) \propto \left(\frac{\mathrm{d}g}{\mathrm{d}x}(x,y,U), \frac{\mathrm{d}g}{\mathrm{d}y}(x,y,U) \right)$$

Note

$$\frac{dg}{dx} = g_x(x,y,U) + g_U(x,y,U)U_x; \qquad \frac{dg}{dy} = g_y(x,y,U) + g_U(x,y,U)U_y$$

Try this:

$$U = F(g(x,y,U)); \qquad g(x,y,U) = F^{-1}(U)$$

In this case you will again have to be satisfied with an implicit relation for $U(x,y)$ in terms of some arbitrary function.

Exercises regarding general solutions:

Finally:

k. Hence or otherwise show that the general solution of the (1+1) PDE

$$v_t + v v_x = 0$$

is given implicitly by

$$v(t,x) = f\Big(x - v(t,x)t\Big).$$

(Challenge: Try to come up with a physical model for a situation where this PDE is relevant.)

Exercises regarding general solutions:

I. Hence or otherwise show that the general solution of the (3+1) PDE

$$\vec{v}_t + (\vec{v} \cdot \nabla)\, \vec{v} = 0$$

is given implicitly by

$$\vec{v}(t, \vec{x}) = \vec{f}\Big(\vec{x} - \vec{v}(t, \vec{x})t\Big).$$

(Challenge: Try to come up with a physical model for a situation where this PDE is relevant.)

Exercises regarding general solutions:

General solution versus singular solution

The definition of general solution for a single first order PDE in a single unknown was that it be a solution involving one arbitrary function.

As for ODEs, the general solution may not always cover all possible solutions (those "extra" solutions are called singular solutions).

See, for example, Courant and Hilbert, volume 2, pp. 2 ff. (S 1).

Here is an example:

Consider the (1+1) PDE

$$\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 2\sqrt{U}$$

Exercises regarding general solutions:

i. Explicitly verify that $U = [x + \eta(x + y)]^2$ is a solution, for any arbitrary function $\eta(\bullet)$.
Therefore, since we have a solution to a first order PDE containing one arbitrary function, this is an example of a "general solution".

ii. Show that $U = 0$ is also a specific solution to the equation.

iii. Show that one cannot express the specific solution $U = 0$ in the form $[x + \eta(x + y)]^2 = 0$ for any function η.
Thus we have found a specific solution that does not follow from the general solution!!

iv. What is the "obvious reason" why this particular example is so odd?
Hint: Try the substitution $U(x, y) = W(x, y)^2$

For a general discussion of singular solutions for such equations see M. J. Hill, Proceedings of the London Mathematical Society, 1917.

Exercises regarding general solutions:

Write down, using whatever technique you find easiest, the general solution for these PDEs:

a. $y\,\dfrac{\partial U}{\partial x} - x\,\dfrac{\partial U}{\partial y} = 0.$

b. $x\,\dfrac{\partial U}{\partial x} + y\,\dfrac{\partial U}{\partial y} = 0.$

c. $x\;U\,\dfrac{\partial U}{\partial x} + y\;U\,\dfrac{\partial U}{\partial y} = xy.$

d. $\tan x\,\dfrac{\partial U}{\partial x} + \tan y\,\dfrac{\partial U}{\partial y} = \tan U.$

e. $y\,\dfrac{\partial U}{\partial x} + z\,\dfrac{\partial U}{\partial y} - x\,\dfrac{\partial U}{\partial z} = 0.$

Exercises regarding general solutions:

Boundary value problems

Solve the following boundary value problems by first obtaining, using that innate cunning for which Math 301 students are renowned, the general solutions of the PDEs and then fitting them to the given boundary conditions:

a. $U_{xx} = \frac{1}{c^2} U_{tt}$, given that $U(x,0) = 0$ and $U_t(x,0) = 1/(1+x^2)$.
b. $U_{xx} = 2xy$, given that $U(0,y) = y^2$ and $U_x(0,y) = y$.
c. $V_{xy} = 1$, given that $V = 0$ and $V_x = 0$ when $x + y = 0$.

Classify these BCs as to whether they are Dirichlet, Neumann, Robin, or something else.

End: