Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs — Autumn 2024

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Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Existence and uniqueness of solutions

Existence and uniqueness of solutions:

Definition

A function $U = U(x, y)$ is a solution of the PDE

$$F\left(x, y, U, U^{(1)}, U^{(2)}, \ldots, U^{(n)}\right) = 0$$

on a region W of the plane IR^2 if:

- $U(x, y)$ and its partial derivatives

  $$U^{(1)}(x, y), \ldots, U^{(n)}(x, y)$$

  exist on W.
- For every (x, y) in W

  $$F\left(x, y, U(x, y), U^{(1)}, U^{(2)}, \ldots, U^{(n)}\right) = 0.$$

Existence and uniqueness of solutions:

That is, $U(x, y)$ can be differentiated as often as necessary, and when substituted back into the PDE it makes the equation true.

Warning

Sometimes solutions in the sense given above are called "classical solutions". (Sometimes "strong solutions".)

Warning

There is a whole separate issue of so-called "weak solutions" of PDEs.
Not appropriate for MATH 301.
See chapter 10 of Olver.

Existence and uniqueness of solutions:

Comment

The general situation regarding existence and uniqueness of solutions for systems of PDEs is considerably more complicated than for ODEs.

Below we give a very cursory description of the situation.

The Cauchy Theorem:

Only in the case where all functions involved in defining the PDE are analytic is there an existence and uniqueness result of complete generality resembling the EUS (Existence and Uniqueness of Solutions) theorem for ODEs.

Analytic = infinitely differentiable and with a Taylor series that has a finite radius of convergence.

The most basic of the EUS theorems, which is easy to state and to understand, and which initiated many of the later developments in the theory of PDEs, is due to Cauchy.

See, for example, Courant and Hilbert, volume 2, pp. 39 ff. (S 7).

The Cauchy Theorem:

Theorem (Cauchy)

Consider the PDE

$$\frac{\partial U}{\partial x} = f\left(x, y, U, \frac{\partial U}{\partial y}\right).$$

This is a first-order PDE in "normal form" with one dependent variable and two independent variables.

. . . . *(continued)*

The Cauchy Theorem:

Theorem (Cauchy, continued)

Consider the initial condition that

$$U(0, y) = g(y),$$

is, at $x = 0$, a prescribed analytic function of the independent variable y.

. . . . *(continued)*

The Cauchy Theorem:

Theorem (Cauchy, continued)

Suppose furthermore that $f(\bullet, \bullet, \bullet, \bullet)$ is an analytic function of its arguments.

Then there exists one, and only one, unique solution satisfying these initial conditions.

The Cauchy Theorem:

- Note that you have to make some very powerful assumptions to be able to derive the theorem — much more powerful than those needed for the EUS (existence and uniqueness theorem) for ODEs.
- You can find a generalized version of the theorem and proof discussed fully in Courant and Hilbert (reference below), [volume 2] pages 39–56.
- Note that you are only trying to solve a first-order PDE, but to derive the theorem you need to make analyticity assumptions for $f(x, y, \cdots)$. That is — infinitely differentiable and a convergent Taylor series.
- So the hypotheses you have to put in are very strong compared to the result you wish to prove.

The Cauchy Theorem:

- Note that the two independent variables x and y are treated asymmetrically.
- Cauchy's theorem can be generalized in a number of ways:
  - To many independent variables, to higher order PDEs, and to systems of PDEs. This is relatively "straightforward" and leads to the Cauchy–Kowalewsky theorem.
  - To more complicated (though still analytic) PDEs — this leads to the Riquier–Janet theory.
  - To many different non-analytic but relatively simple PDEs — these are often the most useful EUS theorems in practice.

Reminders:

Reminder

Analytic, C^ω, means infinitely differentiable and expandable as a power series with non-zero radius of convergence.
Smooth, C^∞, just means infinitely differentiable.
C^2 means twice differentiable [with continuous derivative].
C^1 means once differentiable [with continuous derivative].
C^0 means continuous.

Example

$\exp(x)$ is C^ω for finite x.
$\exp(1/x)$ is C^ω for finite positive x, but not even C^0 at $x = 0$.
$|x^3|$ is C^2 but not C^3.
$|x|$ is C^0 but not C^1.

The Cauchy–Kowalewsky Theorem:

A reasonably well-known generalization of the Cauchy theorem (which is however still a very special case of the Riquier–Janet theory) is the Cauchy–Kowalewsky Theorem, which I quote below for the case of a system of PDEs of the k-th order with several dependent variables U^A, which are functions of the $n+1$ independent variables $x, y^1, y^2, \dots, y^n$.

Note that one of the independent variables, x, has been singled out for special treatment!

(That is, one of the coordinates is treated differently from the others!)

The Cauchy–Kowalewsky Theorem:

Historical note

Since the Russian alphabet is radically different from the English alphabet, and since she published a lot of work in German [and French, and Swedish?], poor Sophie (Sofia, Sonya) Kowalewsky's name has gotten rather mangled over the years.

In addition to Kowalewsky I have seen Kovalevskaya, Kowalevskaya, and Kovalevski.

I'm sure there's other variants out there.

See: http://en.wikipedia.org/wiki/Sofia_Kovalevskaya

The Cauchy–Kowalewsky Theorem:

Historical note

Sophie Kowalewsky (1850-91) did important work in partial differential equations.
Born in Moscow, she married a paleontologist and moved to Germany.
At the University of Heidelberg she studied privately with the great mathematician Weierstrass; women were not allowed at lectures.
She received a degree in absentia in 1874 for her thesis on partial differential equations.
Her most famous work tells conditions when a partial differential equation has a solution that is unique and analytic.
She won the Paris Academy Prize in 1888 for a paper on the integration of the equations of motion for a solid body rotating around a fixed point; the paper was of such high quality that the announced award money was doubled.
In 1889 she became a professor of mathematics at Stockholm.
In addition to her work in mathematics, she wrote some noted novels depicting life in Russia.

Historical note

Courant and Hilbert credit Cauchy with the basic idea for this theorem, and credit Kowalewsky with carrying out the proof "in a rather general manner".

See, for example, Courant and Hilbert, volume 2, pp. 39 ff. (S 7).

The Cauchy–Kowalewsky Theorem:

Theorem (**Cauchy–Kowalewsky**)

Consider the system of PDEs

$$\frac{\partial^k U^A}{(\partial x)^k} = f^A\left(x, y^1, ..., y^n, U^B, \frac{\partial U^B}{\partial x}, \dots, \frac{\partial^{k-1} U^B}{(\partial x)^{k-1}}, \frac{\partial U^B}{\partial y^i}, \dots, \frac{\partial^k U^B}{(\partial y^i)^k}\right).$$

Consider the initial conditions that the functions

$$U^A(0, y^1, \dots, y^n), \quad \frac{\partial U^A}{\partial x}(0, y^1, \dots, y^n), \quad \frac{\partial^2 U^A}{(\partial x)^2}(0, y^1, \dots, y^n),$$

and

$$\frac{\partial^{k-1} U^A}{(\partial x)^{k-1}}(0, y^1, \dots, y^n)$$

are, at $x = 0$, all prescribed analytic functions of the independent variables: y^1, ..., y^n.

. . . . *(continued)*

The Cauchy–Kowalewsky Theorem:

Theorem (Cauchy–Kowalewsky, continued)

Suppose furthermore that the functions $f^A(\bullet, \bullet, \bullet, \cdots)$ are analytic functions of their arguments.

Then there exists one and only one unique solution satisfying these initial conditions.

The Cauchy–Kowalewsky Theorem:

- When the PDE is presented in this manner it is said to be in "normal form".
- Note that this is a k'th order system of PDEs in $(n+1)$ independent variables — that is, defined on a space with $(n+1)$ coordinates.
- The number of equations, and hence the number of dependent variables, is arbitrary.
- Note that the initial conditions are all specified on the very special hyperplane $x = 0$.
- You can find the theorem and proof discussed fully in Courant and Hilbert (reference below), [volume 2] pages 39–56.

The Cauchy–Kowalewsky Theorem:

- Note that the Courant and Hilbert book is definitely not light reading; it is however a gold-mine of highly technical information.
- Note that you are only trying to solve a k'th order system of PDEs, but to derive the theorem you need to make analyticity assumptions for $f(x, \cdots)$. That is — infinitely differentiable and a convergent Taylor series. The hypotheses you have to put in are very strong compared to the result you wish to prove.
- To see what is going on it is convenient to work with systems of first-order PDEs in two independent variables x and y. As Courant and Hilbert say, "there is no modification necessary for more independent variables". Because we are now dealing with systems of first-order PDEs, this is still a significant generalization of the original Cauchy theorem.

Theorem (Cauchy–Kowalewsky (simplified))

Consider the system of PDEs

$$\frac{\partial U^A}{\partial x} = f^A\left(x, y, U^B, \frac{\partial U^B}{\partial y}\right).$$

Consider the initial conditions that

$$U^A(0, y) = g^A(y),$$

are, at $x = 0$, all prescribed analytic functions, $g^A(y)$, of the independent variable y.

. . . . (continued)

Theorem (Cauchy–Kowalewsky (simplified), (continued))

Suppose furthermore that the $f^A(\bullet, \bullet, \bullet, \bullet)$ *are analytic functions of their arguments.*

Then there exists one, and only one, unique solution satisfying these initial conditions.

The Cauchy–Kowalewsky Theorem:

- Courant and Hilbert state:
  To prove the theorem one first formally constructs power series for the solution and then shows the uniform convergence of these series.
- The details are “straightforward” and are left as an exercise for the reader.
- Remember how to translate that code word “straightforward”?

The non-analytic case:

If the PDE involves non-analytic coefficients, or a non-analytic function F relating the partial derivatives, then the situation is not particularly general at all:

- A single first-order PDE in a single unknown, with given ICs, is known to have a unique solution, and methods for its construction are available.
  - That is, equations of the form

    $$F\left(x, y, U^{(1)}, U\right) = 0$$

    are sufficiently simple that EUS theorems can be developed.
  - See, for example, Courant and Hilbert, volume 2, pp. 22 ff. (S 4).

The non-analytic case:

- We can also develop rather simpler EUS theorems for first-order linear equations of the form

$$\sum_{i=1}^{n} a^i(x_1,\dots,x_n)\,\frac{\partial U(x_1,\dots,x_n)}{\partial x^i}$$

$$+b(x_1,\dots,x_n)\;U(x_1,\dots,x_n)+f(x_1,\dots,x_n)=0.$$

- Such an equation can be directly related to a system of first-order ordinary DEs, leading to the theory of "characteristics".
- See Forsyth (reference below), Courant and Hilbert, Hormander (reference below) for more details.
  - See, for example, Courant and Hilbert, volume 2, pp. 28 ff. (S 5).

The non-analytic case:

- Similarly we can also develop rather simple EUS theorems for some first-order quasi-linear equations of the form

$$\sum_{i=1}^{n} a^i(x_1, \dots, x_n, U) \, \frac{\partial U(x_1, \dots, x_n)}{\partial x^i} + f(x_1, \dots, x_n, U) = 0.$$

  This leads to a generalization of the theory of characteristics.

  - See, for example, Courant and Hilbert, volume 2, pp. 28 ff. (S 5).

The non-analytic case:

- For a system of first-order equations in a single unknown, consistency conditions can be formulated, and methods for the construction of the unique solution for given consistent initial conditions have been found — see, for example, Forsyth again.
- (This situation can be transformed into a special case of the Frobenius–Mayer system, as discussed below).
- A general system of first order PDEs in many unknowns is very difficult to analyse, and only special cases are known (see, for example, Forsyth again).

The non-analytic case:

- Warning: You can always take a single n'th-order PDE in one dependent variable, and recast it as a system of n first-order PDEs in n dependent variables.
- However the converse is not true for PDEs (though it is true for ODEs).
- That is:
  - Given a system of n first-order ODEs it is in general possible to reduce this to a single equivalent n'th order ODE.
  - Given a system of n first-order PDEs it is in general not possible to reduce this to a single equivalent n'th order PDE.
  - See, for example, Courant and Hilbert, volume 2, pp. 58 ff. (Appendix 2 to Chapter 1).
- There is no single unified theory of PDEs — it's very much a collection of special cases (some more general than others).

The non-analytic case:

- Hormander, L., Linear Partial Differential Equations,
  Academic Press N.Y. 1963.
- Courant R., and D. Hilbert,
  Methods of Mathematical Physics Vols 1 and 2, Interscience 1966.
- Forsyth R., Differential Equations, in six volumes,
  Oxford University Press, (1906 onwards).

  This opus covers a large number of techniques, many of which are now mostly forgotten, but which crop up from time to time in research papers.

EUS for specific PDEs:

Although, as we have just seen, the general theory of EUS for generic PDEs is quite patchy and relatively ill-developed (compared to EUS for ODEs), the situation for specific PDEs is often (not always) a lot better.

If some specific PDE has become important for some specific physical/ chemical/ biological/ financial/ military or other reason, then there has generally been a lot of hard work done on the EUS problem for that specific PDE.

So in some specific cases we can say a lot, in other cases things are still a bit of a mess.

EUS for specific PDEs:

Exercise

Solve the following first order linear PDE:

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = x \cos(xy)$$

Do this by making a cunning transformation of variables $s = x + y$, $t = x - y$ *and rewriting the equation in terms of these variables.*

EUS for specific PDEs:

Challenge:
Read and understand the theory of characteristic curves.

Challenge:
Read and understand the proof of the Cauchy theorem.

Challenge:
Read and understand some advanced books on PDEs.

Challenge:
Find, read, and understand some recent PhD theses on PDEs.

EUS for specific PDEs:

I reiterate:

- PhD theses are still being written on (advanced) first-order systems of PDEs.
- PhD theses are still being written on (advanced) second-order PDEs.
- PhD theses are still being written on the general theory of PDEs.

End: