Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Frobenius–Mayer systems

Frobenius–Mayer systems:

Frobenius–Mayer systems are a specific example of a system of PDEs that is sufficiently simple to enable us to obtain a EUS theorem without having to make analyticity assumptions.

Frobenius–Mayer systems:

Definition

Frobenius/Mayer system:

One special case that is very important is the **Frobenius** or **Mayer** system

$$\frac{\partial U^A}{\partial x^i} = F^A{}_i(x^1, \ldots, x^n, U^1, \ldots, U^m) \qquad (F)$$

$$A = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n$$

where the m dependent variables $\{U^A\}$ depend on the n independent variables $\{x^i\}$.

Frobenius–Mayer systems:

Definition (continued)

Frobenius/Mayer system:

All these equations are of first order.

In such a system there are as many PDEs as there are first-order derivatives of the dependent functions (i.e., nm of them).

Notes:

- We see that the Frobenius–Mayer PDE systems are examples of first-order quasi-linear PDE systems.
- The superscripts now tell you which of the U's you are dealing with; not the order of the derivative.
- The only derivatives occurring above are first-order on the LHS. (And they occur linearly with coefficient unity.)
- The RHS of the system does not involve any derivatives.

Frobenius–Mayer systems:

- Just because it's important does not mean it's easy to find any explicit discussion of this system.
- You can find a discussion in Volume 1 of Spivak, chapter 6.
  See especially pages 254–257.
  (The notation is slightly different).
- You can find a discussion in Volume 5 of Forsyth, chapter 4.
  See especially pages 100 ff.
  (The notation is, unfortunately, seriously archaic).

Frobenius–Mayer systems:

- Courant R., and D. Hilbert, Methods of Mathematical Physics Vols 1 and 2, Interscience 1966.
- Forsyth R., Differential Equations, in six volumes, Oxford University Press, (1906 onwards).
- Spivak, M., A comprehensive introduction to differential geometry, in six volumes, (Publish or Perish, Berkeley, 1979).

Integrability theorem:

Theorem (Frobenius Complete Integrability Theorem)

Suppose the functions ${F^A}_i(\bullet, \dots)$ are smooth C^1 functions of all their variables in a neighbourhood of the origin, for $A = 1, 2, \dots, m$, and $i = 1, 2, \dots, n$.
Then the Frobenius system (F) has a unique solution satisfying the IC

$$U^A(0, 0, \dots, 0) = b^A \qquad (A = 1, 2, \dots, m)$$

for arbitrary given b^A, if and only if

$$\frac{\partial {F^A}_i}{\partial x^j} + \sum_{B=1}^{m} {F^B}_j \frac{\partial {F^A}_i}{\partial U^B} = \frac{\partial {F^A}_j}{\partial x^i} + \sum_{B=1}^{m} {F^B}_i \frac{\partial {F^A}_j}{\partial U^B} \qquad (C)$$

for all i, j, and A in their respective ranges.

Integrability theorem:

- Note that we only require F to be C^1 instead of C^ω.
- That C^1 is a necessary condition is obvious — it is required so that the relevant derivatives in the compatibility condition (C) exist.
- This Frobenius integrability theorem is an extremely important result.
- The condition (C) is effectively the requirement that the second partial derivatives should all commute:

$$\frac{\partial^2 U^A}{\partial x^i \, \partial x^j} = \frac{\partial^2 U^A}{\partial x^j \, \partial x^i}.$$

Integrability theorem:

- To see necessity (not sufficiency) note that if the PDE defining the Frobenius–Mayer system is satisfied, then

$$\frac{\partial^2 U^A}{\partial x^i\, \partial x^j} = \frac{\mathrm{d}}{\mathrm{d}x^i} F^A_j(x, U(x))$$

Then by applying the chain rule

$$\frac{\partial^2 U^A}{\partial x^i\, \partial x^j} = \frac{\partial}{\partial x^i} F^A_j + \sum_{B=1}^{m} \frac{\partial F^A_j}{\partial U^B}\, \frac{\partial U^B}{\partial x^i}.$$

Now use the Frobenius–Mayer PDE again, we see

$$\frac{\partial^2 U^A}{\partial x^i\, \partial x^j} = \frac{\partial}{\partial x^i} F^A_j + \sum_{B=1}^{m} \frac{\partial F^A_j}{\partial U^B}\, F^B_i.$$

But the LHS is symmetric under interchange $i \longleftrightarrow j$.
This leads to the consistency condition (C).

Integrability theorem:

Consistency condition:

$$\frac{\partial F^A{}_i}{\partial x^j} + \sum_{B=1}^{m} F^B{}_j \frac{\partial F^A{}_i}{\partial U^B} = \frac{\partial F^A{}_j}{\partial x^i} + \sum_{B=1}^{m} F^B{}_i \frac{\partial F^A{}_j}{\partial U^B} \qquad (C)$$

- You can find a full proof, [both necessity and sufficiency], in Volume 1 of Spivak, chapter 6, pages 254–257.
- Note that Spivak's notation is slightly different.

Integrability theorem:

- You can get a feel for how important the Frobenius integrability theorem is from Spivak's comment:

  > *The Frobenius theorem (which represents everything we know about partial differential equations) was used in [...long list of topics...].*

  (See Spivak, volume 5, page 1).

Integrability theorem:

- This should be balanced against his further comment:

  Now it's really rather laughable to call these things partial differential equations at all. True ... partial derivatives are involved, but we do not posit any relationship between different partial derivatives; this comes out quite clearly in the proof [of the integrability theorem] where the equations are reduced to ordinary differential equations.

Proof of integrability theorem:

Proof (not examinable).

Consider, in the specified coordinate chart, the "straight line"

$$x^i(t) = t\, x^i$$

and on this "straight line" for fixed x solve (with respect to t) the ODE:

$$\frac{\mathrm{d}W^A(x;t)}{\mathrm{d}t} = x^i\; F^A{}_i(t\, x^i; W^B(x;t)); \qquad W^A(x;0) = b^A. \tag{1}$$

Since this is simply an ODE, (albeit a non-autonomous coupled ODE in m variables), it will have unique solutions, at least on some finite interval. Now use the $W^A(x;t)$ to define quantities $U^A(x^i)$ as follows

$$U^A(x^i) = W^A(x^i, 1) = b^A + \sum_j x^j \int_0^1 F^A{}_j(t\, x^i; W^A(x;t))\, \mathrm{d}t. \tag{2}$$

These $U^A(x^i)$ certainly exist, but what PDEs do they satisfy? □

Proof of integrability theorem:

Proof (continued):

From

$$U^A(x^i) = b^A + \sum_j x^j \int_0^1 F^A{}_j(t\,x^i; W^A(x;t))\,\mathrm{d}t, \tag{3}$$

let us compute

$$\begin{aligned}
\partial_i U^A(x) \;&=\; \int_0^1 F^A{}_i(t\,x; W(x;t))\,\mathrm{d}t \\
&\quad + \sum_j x^j \int_0^1 t\{\partial_i F^A{}_j(t\,x; W(x;t)) \\
&\qquad\qquad + \partial_B F^A{}_j(t\,x; W(x;t))\, \partial_i W^B(x;t)\}\,\mathrm{d}t.
\end{aligned} \tag{4}$$

□

Proof of integrability theorem:

Proof (continued):

But that first term can be integrated by parts as

$$
\begin{aligned}
\int_0^1 F^A{}_i(t\,x; W(x;t))\,\mathrm{d}t &= [tF^A{}_i(t\,x; W(x;t))]_0^1 \\
&\quad - \int t\frac{\mathrm{d}}{\mathrm{d}t}[F^A{}_i(t\,x; W(x;t))]\mathrm{d}t \\
&= F^A{}_i(x; W(x,1)) \\
&\quad - \int_0^1 t\left\{F^A{}_{i,j}(t\,x; W(x,t))x^j + \partial_B F^A{}_i \dot{W}^B\right\}\mathrm{d}t \\
&= F^A{}_i(x; U(x)) \\
&\quad - \int_0^1 t\left\{F^A{}_{i,j}(t\,x; W(x,t))\,x^j + \partial_B F^A{}_i F^B_j x^j\right\}\mathrm{d}t.
\end{aligned}
$$

Proof of integrability theorem:

Proof (continued):

Combining

$$\partial_i U^A(x) \quad = \quad \int_0^1 F^A{}_i(t\,x; W(x;t))\,\mathrm{d}t + x^j \int_0^1 t\{\partial_i F^A{}_j(t\,x; W(x;t)) + \partial_B F^A{}_j(t\,x; W(x;t))\,\partial_i W^B(x;t)\}\,\mathrm{d}t,$$

and

$$\int_0^1 F^A{}_i(t\,x; W(x;t))\,\mathrm{d}t \quad = \quad F^A{}_i(x; U(x)) - \int_0^1 t\left\{F^A{}_{i,j}(t\,x; W(x,t))\,x^j + \partial_B F^A{}_i F^B_j x^j\right\}\mathrm{d}t,$$

we have:

$$\partial_i U^a(x) = F^A{}_i(x; U(x)) + \sum_j x^j \int_0^1 t\{F^A{}_{i,j} - F^A{}_{j,i} + \partial_B F^A_j \partial_i U^B - \partial_B F^A_i F^B_j\}\mathrm{d}t.$$

Now apply the consistency condition.

□

Proof of integrability theorem:

Proof (continued):

Now apply the consistency condition.
Thence

$$\partial_i U^a(x) - F^A{}_i(x; U(x)) = \int_0^1 t \sum_j x^j \partial_B F_j^A \; \left\{\partial_i U^B(tx) - F_i^B(tx, U(tx))\right\} \mathrm{d}t.$$

This is an integral equation.

One solution is clearly $\partial_i U^a(x) = F^A{}_i(x; U(x))$.

□

Proof of integrability theorem:

Proof (continued):

As long as the integral transform does not have eigenvalue unity, this will be the only solution.

(And for sufficiently small x, where the integral transform is guaranteed to be small, this will certainly be the unique solution.)

So under the stated consistency condition we have established the existence of a set of fields $U^A(x^i)$ such that $\partial_i U^a(x) = F^A{}_i(t\,x; U(x))$.

□

(This proof is slightly different from other presentations you might eventually track down, either on the internet or in various older texts. I feel the present discussion is pedagogically simpler.)

Integrability theorem:

Comments:

- Clearly if $n = 1$ [only one independent variable, one dimension] then condition (C) is always satisfied.
- But this just means that if we have one independent variable then the 1-dimensional Frobenius equation

$$\frac{\partial U^A}{\partial x} = F^A(x, U^1, \ldots, U^m) \qquad A = 1, 2, \ldots, m, \qquad (1d\ F)$$

  is always integrable.
- This will be less of a surprise if we realise this is now an ODE, and change variables ($x \to t$, $U^A \to x^A$) to rewrite it in the more usual form

$$\frac{\mathrm{d}x^A}{\mathrm{d}t} = F^A(t, x^B) \qquad A = 1, 2, \ldots, m.$$

- We already know, by elementary means, that this simple ODE is integrable.

Integrability theorem:

- A second important case is $m = 1$ [only one dependent variable, one "field" but many dimensions] then condition (C) reduces to

$$\frac{\partial F_i}{\partial x^j} + F_j \frac{\partial F_i}{\partial U} = \frac{\partial F_j}{\partial x^i} + F_i \frac{\partial F_j}{\partial U} \qquad (1 \text{ variable } C)$$

- That is

$$\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i} + F_j \frac{\partial F_i}{\partial U} - F_i \frac{\partial F_j}{\partial U} = 0 \qquad (1 \text{ variable } C)$$

- Alternatively

$$\partial_j F_i - \partial_i F_j + F_j \frac{\partial F_i}{\partial U} - F_i \frac{\partial F_j}{\partial U} = 0 \qquad (1 \text{ variable } C)$$

- This is one of the most common cases to arise in practice.

- It is sometimes useful to rewrite condition (C) in the equivalent form

$$\frac{\partial F^A{}_i}{\partial x^j} - \frac{\partial F^A{}_j}{\partial x^i} = \sum_{B=1}^{m} \left\{ F^B{}_i \, \frac{\partial F^A{}_j}{\partial U^B} - F^B{}_j \, \frac{\partial F^A{}_i}{\partial U^B} \right\} \qquad (C)$$

- Alternatively

$$\partial_j F^A{}_i - \partial_i F^A{}_j = \sum_{B=1}^{m} \left\{ F^B{}_i \, \frac{\partial F^A{}_j}{\partial U^B} - F^B{}_j \, \frac{\partial F^A{}_i}{\partial U^B} \right\} \qquad (C)$$

- Doing this should focus your attention on conservative vector fields as a possible way of satisfying the integrability constraints.

Integrability theorem:

- A sufficient condition for condition (C) to hold in general is that

$$F^A{}_i(x,U) = \frac{\partial \Phi(x)}{\partial x^i}\, J^A(U); \qquad (C2)$$

- Try it and see. (I do not claim this condition is necessary.)
- If this sufficient condition holds then the Frobenius/ Mayer system reduces to

$$\frac{\partial U^A}{\partial x^i} = \frac{\partial \Phi(x)}{\partial x^i}\, J^A(U).$$

- But now we can solve this by reducing it to an ODE. (continued)

Integrability theorem:

- (continued)   Note that each of the U^A, considered as a function of the x^i, can change only in the direction parallel to

$$\partial_i \Phi(x) = \frac{\partial \Phi(x)}{\partial x^i}.$$

- But this means that for some set of functions $\widetilde{U}^A(\Phi)$ we have

$$U^A(x) = \widetilde{U}^A(\Phi(x)),$$

with the PDE reducing to

$$\frac{\mathrm{d}\tilde{U}^A(\Phi)}{\mathrm{d}\Phi} = J^A(\tilde{U}).$$

- This reduces the Frobenius/ Mayer system,
[subject to this sufficient condition ($C2$)], to an ODE.
- In fact it is an autonomous ODE, which we already know to be integrable.

Integrability theorem:

- There is an even more special case, obvious given the above discussion, that I will belabour because of its importance:

— the autonomous Frobenius/ Mayer system. —

Autonomous Frobenius–Mayer systems

Autonomous Frobenius–Mayer systems:

Definition

The autonomous **Frobenius**/ **Mayer** system is

$$\frac{\partial U^A}{\partial x^i} = F^A{}_i(U^1, \ldots, U^m)$$

$$A = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n \quad (AF)$$

Autonomous Frobenius–Mayer systems:

- Note: The key feature is that there is now no explicit x dependence on the RHS.
- The class of autonomous Frobenius/ Mayer systems can be characterized as a particular sub-class of autonomous first-order quasi-linear PDEs.
- The m dependent variables $\{U^A\}$ again depend on the n independent variables $\{x^i\}$.
- All these equations are again of first order.
- There are again as many PDEs as there are first-order derivatives.
- That is, nm of them.

Autonomous Frobenius–Mayer systems:

- The RHS now depends only on the dependent variables, the U's.
- There is no explicit x dependence on the RHS.
- The equations are "autonomous" in the sense that the "driving term" does not pay any attention to the independent variables, the x's.
- The "driving term" or "source term" now depends only on the "current state" of the system — the U's.

The Autonomous Frobenius Integrability Theorem:

Theorem (**Autonomous Frobenius Integrability**)

Suppose the functions $F^A{}_i(U^A)$ are smooth C^1 functions of all their variables in a neighbourhood of the origin, for $A = 1, 2, \ldots, m$.

Then the autonomous Frobenius system (AF) has a unique solution, satisfying the IC

$$U^A(0, 0, \ldots, 0) = b^A \qquad (k = 1, 2, \ldots, m)$$

for arbitrary given b^A, if and only if

$$\sum_{B=1}^{m} F^B{}_i \; \frac{\partial F^A{}_j}{\partial U^B} = \sum_{B=1}^{m} F^B{}_j \; \frac{\partial F^A{}_i}{\partial U^B} \qquad (AC)$$

for all i, j, and A in their respective ranges.

But now let's take a more careful look at the condition (AC).

- If $n = 1$, [so that we are working in one dimension], condition (AC) is always satisfied.
  But this is just the autonomous version of our previous discussion.
  After a change in notation ($x \to t$, $U^A \to x^A$) the 1-d autonomous Frobenius equation becomes

  $$\frac{\mathrm{d}x^A}{\mathrm{d}t} = F^A(x^B) \qquad\qquad A = 1, 2, \ldots, m.$$

  This always has a solution...

The Autonomous Frobenius Integrability Theorem:

- Suppose in contrast that $m = 1$ so there is only one dependent variable U, only a single "field".
  Then condition (AC) reduces to

$$F_i \frac{\partial F_j}{\partial U} = F_j \frac{\partial F_i}{\partial U} \qquad (1 \text{ variable } AC)$$

  But this is satisfied iff (if and only if) $F_i/F_j = k_i/k_j$ for some set of constants k_i independent of U.
  That implies

$$F_i(U) = k_i \, f(U)$$

  for some constant vector k_i.

(continued)

- (continued)
  But this now lets us write the 1-variable integrable autonomous Frobenius system as

$$\frac{\partial U}{\partial x^i} = k_i \, f(U) \qquad\qquad i = 1, 2, \ldots, m.$$

Thus the system (if it satisfies condition (AC) so that it is integrable) can be reduced to an ODE in a single variable, call it ξ:

$$U(x) = \tilde{U}\,(k \cdot x)\,; \qquad \frac{\mathrm{d}\tilde{U}(\xi)}{\mathrm{d}\xi} = f(\tilde{U})$$

Note that this is all a special case of condition (C2) above.
This now always has a solution...

The Autonomous Frobenius Integrability Theorem:

- In fact for any n and m, a sufficient condition for condition (AC) to hold is that
$$F^A{}_i(x, U) = k_i \; J^A(U); \qquad (AC2)$$
- Try it and see. (I do not claim this condition is necessary.)
- If this sufficient condition holds then the autonomous Frobenius–Mayer system reduces to
$$\frac{\partial U^A}{\partial x^i} = k_i \; J^A(U).$$
- But we can again solve this by reducing it to an ODE.
- Note that each of the U^A, considered as a function of the x^i, can change only in the direction parallel to k_i.

(continued)

- (continued)
  But this means that for some set of functions $\widetilde{U}^A(\xi)$ we have

  $$U^A(x) = \widetilde{U}^A(\xi); \qquad \xi = \xi_0 + \sum_{i=1}^{m} k_i \, x^i$$

  with the PDE reducing to

  $$\frac{\mathrm{d}\tilde{U}^A(\xi)}{\mathrm{d}\xi} = J^A(\tilde{U}).$$

- This again reduces the autonomous Frobenius–Mayer system [subject to this sufficient condition (AC2)] to an ODE.

The Autonomous Frobenius Integrability Theorem:

- Clearly the most "interesting" cases are $n > 1$ and $m > 1$.
- You can have some fun exploring necessary and sufficient conditions, and digging deep into the bowels of the library.

Conservative vector fields:

A vector field V is called conservative if curl $V = 0$.
It is a well known fact that if V is conservative on a (topologically trivial) open subset W of $I\!R^3$, then there is a function $U(x, y, z)$ such that

$$\vec{V} = -\text{grad } U$$

on W.

We now want to relate this to the concept of a Frobenius–Mayer system.

Conservative vector fields:

Exercise:

a. Show that the system of PDEs

$$\text{grad } U = -\vec{V}$$

is a Frobenius system, (a particularly simple Frobenius system).
Furthermore, show that it can be made to satisfy the conditions of the Frobenius Complete Integrability theorem.
Explicitly find the consistency condition.

b. Find the function U if:
   i. $\vec{V} = xyz\,\vec{i} + (x^2 z/2 - z\sin(yz))\,\vec{j} + (x^2 y/2 - y\sin(yz))\,\vec{k}$.
   ii. $\vec{V} = (A/r^3)\,\vec{r}$.
   Here A is a constant, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ is the usual radius vector $\vec{r}$, and $r = |r|$.

Exercises and examples

Exercise:
(A slightly more complicated example; essentially two-dimensional)

Consider now a specific Frobenius theorem with $m = 1$ (so there is only one dependent variable, which I will call h) and $n = 2$ (so there are two independent variables, two dimensions, which I shall call x and y).

Then the Frobenius system is

$$\frac{\partial h(x,y)}{\partial x} = F_x(x,y,h)$$

$$\frac{\partial h(x,y)}{\partial y} = F_y(x,y,h)$$

Height-slope relations:

Exercise (continued):

You can interpret this, for instance, as the equation for the height of a hill as a function of x and y, given that there is a PDE controlling the height of the hill which makes the slope of the hill depend on its height,
(a self-referential height-slope function).

Height-slope relations:

Exercise (continued):

a. Explicitly write out the set of consistency conditions required for this Frobenius system to have a solution.

b. Ignoring trivial re-labellings, how many non-trivial consistency conditions are there?

Exercise (continued):

c. Now consider the three-dimensional vector

$$\vec{v}(x,y,z) = \Big(F_x(x,y,z),\ F_y(x,y,z),\ 1\Big)$$

where now I have relabelled $h \to z$.

d. Calculate the "vorticity":

$$\vec{\omega} = \text{curl } \vec{v} = \nabla \times \vec{v}.$$

e. Calculate the "helicity":

$$H = \vec{v} \cdot (\text{curl } \vec{v}) = \vec{v} \cdot (\nabla \times \vec{v}).$$

Exercise (continued):

f. Show that the condition that the helicity vanishes,

$$H = \vec{v} \cdot (\text{curl } \vec{v}) = 0,$$

is equivalent to the Frobenius consistency condition in part [a].

- This implies that if the helicity H of $\vec{v}(x, y, z)$ is zero then it is possible to self-consistently find a height function $z(x, y)$ with

$$\partial_i z(x, y) = v_i(x, y, z).$$

- (This result as given is special to $m = 1$, $n = 2$; there is a generalization of this result to $m = 1$, $n \geq 3$ which is a little tricker to formulate nicely.)

Autonomous example:

Exercise:

(Fully three-dimensional example)

Consider the system of PDEs

$$\partial_x U = h_x(U(x, y, z))$$

$$\partial_y U = h_y(U(x, y, z))$$

$$\partial_z U = h_z(U(x, y, z))$$

Autonomous example:

(Fully three-dimensional example)

1. Write down all the Frobenius integrability conditions for this system. How many of the constraints are nontrivial?
2. By adopting the notation

$$\vec{H} = (h_x, h_y, h_z)$$

   show that the integrability conditions are equivalent to

$$\vec{H} \times \frac{d\vec{H}}{dU} = 0$$

3. Hence show that this system satisfies the integrability conditions iff

$$\vec{H} = \vec{k}\, f(U)$$

   where $\vec{k}$ is a constant vector.

Autonomous example:

4. Finally, show that in this situation the solution of the Frobenius system is given by the implicit equation

$$\int_{U_0}^{U} \frac{\mathrm{d}\bar{U}}{f(\bar{U})} = \vec{k} \cdot \vec{x}.$$

5. That is, show that there exists an invertible function $g(U)$ such that

$$g(U) = \vec{k} \cdot \vec{x},$$

and so

$$U(x) = g^{-1}\left(\vec{k} \cdot \vec{x}\right).$$

6. Indeed, show that

$$\frac{\mathrm{d}g}{\mathrm{d}U} = \frac{1}{f(U)}.$$

Challenges:

- **Challenge:**
  Look up, read, and understand, various other versions of the proof of the Frobenius–Meyer integrability theorem.

- **Challenge:**
  Look up, read, and understand, the connection between the Frobenius–Meyer integrability theorem for PDEs and the "Frobenius theorem" of differential geometry.

End: