Victoria University of Wellington

*Te Whare Wānanga o te Ūpoko o te Ika a Maui*

# — MATH 301 — PDEs —
# Autumn 2024

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21 February 2024

# Outline:

1. Administrivia
2. Other first-order PDEs
3. The continuity equation
4. The hydrodynamic Euler equation

# Administrivia

# Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

# Other first-order PDEs

# Other first-order PDEs:

We have already seen several examples of reasonably general classes of first-order PDEs:

- First-order quasi-linear PDEs:

$$\alpha(x,y,U)\,\partial_x U + \beta(x,y,U)\,\partial_y U + \gamma(x,y,U) = 0.$$

- The PDE of Cauchy's theorem:

$$\frac{\partial U}{\partial x} = f\left(x, y, U, \frac{\partial U}{\partial y}\right).$$

- Frobenius–Mayer systems:

$$\partial_i U^A = F_i^A(x^j, U^B).$$

Two other first-order PDEs of considerable importance are briefly discussed below.

# The continuity equation

# Continuity equation:

The continuity equation is

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

- Used wherever there is a "conservation law" for either mass/ charge/ probability.
- Fluid dynamics (mass).
- Electromagnetism (charge).
- Probabilistic modelling; stochastic equations. (Physics, Statistics, Finance, Biology, Chemistry, Geology.)

# Continuity equation:

In the general situation you would want to think of the velocity $\vec{v}(t, \vec{x})$ as three additional dependent variables, so that you have

$$\partial_t \rho + (\vec{v} \cdot \vec{\nabla})\rho + \rho\,(\vec{\nabla} \cdot \vec{v}) = 0.$$

This is then a first-order quasi-linear PDE connecting (in three space dimensions) four dependent variables.

# The hydrodynamic Euler equation

# Hydrodynamic Euler equation:

The hydrodynamic Euler equation is

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho}$$

This is actually Newton's second law $\vec{F} = m\,\vec{a}$ rewritten in terms of individual little blobs of fluid.

Here $p$ is the pressure, $\vec{B}$ is any external force (for example, gravity).

# Hydrodynamic Euler equation:

For a velocity field $\vec{v}(t,\vec{x})$ the velocity of an individual particle at point $\vec{x}$ at time $t$ is:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \equiv \vec{v}(t,\vec{x})$$

But now take a look at the acceleration:

$$\vec{a} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\vec{v}(t,\vec{x}(t))}{\mathrm{d}t}$$

But by the chain rule

$$\vec{a} = \partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}$$

Note the nonlinearity in the velocity field.
The hydrodynamic Euler equation is another example of a first-order quasi-linear PDE connecting many dependent variables.

# Hydrodynamic Euler equation:

## Exercise

*Suppose the pressure is identically zero, a situation generally referred to as "dust", and there are no body forces.*
*Then Euler's hydrodynamic equation reduces to*

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{0}.$$

*By whatever means you can, demonstrate that this PDE has the (implicit) general solution:*

$$\vec{v}(t,x) = \vec{f}\Big(\vec{x} - \vec{v}(t,\vec{x})t\Big).$$

*Here $\vec{f}(\vec{x})$ is an arbitrary function $IR^n \to IR^n$.*

# Hydrodynamic Euler equation:

- In the next chapter we will use the phrase “Euler equation” in a very different way.
- The nomenclature, with Euler's name being attached to two such very distinct equations, is unfortunately standard.

# End: