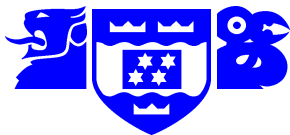


Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui



— MATH 301 — PDEs —
Autumn 2024

Matt Visser

21 February 2024

Outline:

- 1 Administrivia
- 2 Other first-order PDEs
- 3 The continuity equation
- 4 The hydrodynamic Euler equation



Administrivia



- **Lectures:**
 - Monday; 12:00–12:50; MYLT 102.
 - Tuesday; 12:00–12:50; MYLT 220.
 - Friday; 12:00–12:50; MYLT 220.
- **Tutorial:**
 - Thursday; 12:00–12:50; MYLT 220.
- **Lecturers:**
 - Part 1: Matt Visser.
 - Part 2: Dimitrios Mitsotakis.





Other first-order PDEs

Other first-order PDEs:

We have already seen several examples of reasonably general classes of first-order PDEs:

- First-order quasi-linear PDEs:

$$\alpha(x, y, U) \partial_x U + \beta(x, y, U) \partial_y U + \gamma(x, y, U) = 0.$$

- The PDE of Cauchy's theorem:

$$\frac{\partial U}{\partial x} = f\left(x, y, U, \frac{\partial U}{\partial y}\right).$$

- Frobenius–Mayer systems:

$$\partial_i U^A = F_i^A(x^j, U^B).$$

Two other first-order PDEs of considerable importance are briefly discussed below.



The continuity equation

Continuity equation:

The continuity equation is

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

- Used wherever there is a “conservation law” for either mass/ charge/ probability.
- Fluid dynamics (mass).
- Electromagnetism (charge).
- Probabilistic modelling; stochastic equations.
(Physics, Statistics, Finance, Biology, Chemistry, Geology.)

Continuity equation:

In the general situation you would want to think of the velocity $\vec{v}(t, \vec{x})$ as three additional dependent variables, so that you have

$$\partial_t \rho + (\vec{v} \cdot \vec{\nabla}) \rho + \rho (\vec{\nabla} \cdot \vec{v}) = 0.$$

This is then a first-order quasi-linear PDE connecting (in three space dimensions) four dependent variables.



The hydrodynamic Euler equation

Hydrodynamic Euler equation:

The hydrodynamic Euler equation is

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho}$$

This is actually Newton's second law $\vec{F} = m \vec{a}$ rewritten in terms of individual little blobs of fluid.

Here p is the pressure, \vec{B} is any external force (for example, gravity).

Hydrodynamic Euler equation:

For a velocity field $\vec{v}(t, \vec{x})$ the velocity of an individual particle at point \vec{x} at time t is:

$$\frac{d\vec{x}}{dt} \equiv \vec{v}(t, \vec{x})$$

But now take a look at the acceleration:

$$\vec{a} = \frac{d^2\vec{x}}{dt^2} = \frac{d\vec{v}(t, \vec{x}(t))}{dt}$$

But by the chain rule

$$\vec{a} = \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v}$$

Note the **nonlinearity** in the velocity field.

The hydrodynamic Euler equation is another example of a first-order quasi-linear PDE connecting many dependent variables.

Hydrodynamic Euler equation:

Exercise

Suppose the pressure is identically zero, a situation generally referred to as “dust”, and there are no body forces.

Then Euler’s hydrodynamic equation reduces to

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \vec{0}.$$

By whatever means you can, demonstrate that this PDE has the (implicit) general solution:

$$\vec{v}(t, \vec{x}) = \vec{f}(\vec{x} - \vec{v}(t, \vec{x})t).$$

Here $\vec{f}(\vec{x})$ is an arbitrary function $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Hydrodynamic Euler equation:

- In the next chapter we will use the phrase “Euler equation” in a very different way.
- The nomenclature, with Euler’s name being attached to two such very distinct equations, is unfortunately standard.



End:

