Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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Outline:

1. Administrivia
2. Euler Equation: Standard examples
   - Wave equation
   - Heat equation (diffusion equation)
   - Laplace equation
   - Review: Elliptic/ Parabolic/ Hyperbolic
3. Other standard Euler PDEs
   - Klein–Gordon equation
   - Helmholtz equation
   - Maxwell equations
4. Summary

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Euler Equation: Standard examples

Euler Equation: Standard examples:

I'll now give a catalogue of standard examples of Euler PDEs that you should learn to recognize.

(Typical of a hyperbolic PDE).

$$U_{xx} - \frac{1}{c^2}\, U_{tt} = 0.$$

- Here, $U(x,t)$ represents the displacement at point x and at time t of a string from its equilibrium position.
- That is, $U(x,t)$ is the shape of the string at time t.
- The constant c is the velocity of the wave disturbance.
- The same equation can be used to describe sound waves or light waves; at least in flat spacetime.
- The generalizations to get to curved spacetime are not too onerous, but not appropriate for Math 301.

The Wave Equation:

- Usually you know:
  - That the string is fixed at the origin $x = 0$ and at the end-point ($x = L$, say).
  - The initial shape of the string, $U(x, 0)$.
  - The velocity of each point x of the string, $U_t(x, 0)$.
    [most often this will be zero, the string will start from rest].
  - Conditions of this sort, where you know initial values of the function and its derivatives, are called Cauchy (initial) conditions.
  - It can be shown that Cauchy initial conditions are necessary and sufficient for the existence and uniqueness of solutions.
  - (This is typical of those problems that are classified as hyperbolic — Cauchy conditions are enough to guarantee existence and uniqueness of solutions).

The Wave Equation:

- In terms of the Euler PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = 0$$

the wave equation corresponds to

$$a \to 1; \quad h \to 0; \quad b \to -\frac{1}{c^2}$$

with the notational change $y \to t$.

- Without further calculation we can use the analysis of the Euler PDE to immediately write down the general solution of the wave equation:

$$U(x,t) = f(x - ct) + g(x + ct)$$

This is d'Alembert's solution, and I'll have considerably more to say about it later.

The Heat or Diffusion equation

(Typical of a parabolic PDE).

$$\boxed{U_t = \sigma\ U_{xx}.}$$

- Here σ is a constant, called the thermal diffusivity (heat equation) or simply the diffusion constant.
- Such an equation often occurs in situations where diffusion occurs.
- For example, consider a heated bar of metal:
- $U(x,t)$ is the temperature at time t at a point x along the bar.

The Heat Equation:

- You might be given:
  - Ohe initial distribution of temperature in the bar, $U(x, 0)$.
  - Or, you might be told that the two ends of the bar are kept a fixed temperatures,

$$U(0, t) = T_1$$
$$U(L, t) = T_2$$

    where L is the length of the bar.

  Then again you might be told:
  - The initial distribution of temperature in the bar, $U(x, 0)$.
  - Or, you might be told that the ends are insulated, so that no heat can pass through them:

$$U_x(0, t) = 0 = U_x(L, t) \quad \text{for all } t.$$

- Typically, for parabolic equations, conditions of the type described above will guarantee the existence and uniqueness of a solution.
- In terms of the generalized Euler PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} + c\ U_x + d\ U_y + e\ U + f = 0$$

the heat equation corresponds to

$$a \to \sigma; \quad h \to 0; \quad b \to 0;$$

$$c \to 0; \quad d \to -1; \quad e \to 0; \quad f \to 0$$

with the notational change $y \to t$.

The Heat Equation:

- There is no closed-form general solution in terms of algebraic combinations of arbitrary functions.
- But we will later in the course use Fourier transforms to give a general solution in terms of an infinite series of “basis functions”.

Hint:

Both

$$u(x,t) = \cos(kx)\exp(-\sigma k^2 t)$$

and

$$u(x,t) = \sin(kx)\exp(-\sigma k^2 t)$$

solve the heat equation for arbitrary values of k.

Then consider:

$$u(x,t) = \int A(k)\cos(kx)\exp(-\sigma k^2 t)\,\mathrm{d}k + \int B(k)\sin(kx)\exp(-\sigma k^2 t)\,\mathrm{d}k$$

There are two arbitrary functions, but now "hidden" inside the integrals...

The Laplace equation

(Typical of an elliptic PDE).

$$U_{xx} + U_{yy} = 0.$$

- Now $U(x, y)$ represents, for example,
  - the electrostatic potential at the point (x, y) in a piece R of dielectric medium,
  - or the Newtonian gravitational potential in empty space (outside the sources),
  - or it might represent the equilibrium temperature at the point (x, y) inside a heated solid R.

- Typically, in problems involving Laplace's equation, boundary conditions of the following form are known:
  1. You might be given the potential (temperature) on the boundary $B = \partial R$ of the region R:
  $$U(x, y) \quad \text{is given on } B.$$
  Such a condition is called a Dirichlet condition.
  2. You might know the flux of U, (that is, the gradient of U normal to the boundary B), into the region R :
  $$\frac{\partial U}{\partial n} \quad \text{is given on } B.$$
  Such a condition is called a Neumann condition.
  3. Frequently, you might be given a mixture of Dirichlet and Neumann conditions. (Robin boundary conditions.)

The Laplace Equation:

- So long as the boundary shape B is "reasonable", you can be sure there will be a unique solution to Laplace's equation satisfying any of these boundary conditions.
- In terms of the Euler PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = 0$$

the Laplace equation corresponds to

$$a \to 1; \quad h \to 0; \quad b \to 1.$$

The Laplace Equation:

- Without further calculation we can use the analysis of the Euler PDE to immediately write down the general solution of Laplace's equation:

$$U(x, y) = f(x + iy) + g(x - iy)$$

- This is Laplace's solution, which relates the solution of the Laplace PDE to the theory of functions of a complex variable.
- I'll also have more to say about this later.

Review: Elliptic/ Parabolic/ Hyperbolic

Euler PDE versus Laplace PDE:
When is the Euler differential equation elliptic?
When is the Euler differential equation qualitatively similar to Laplace's equation?
When is it qualitatively different?

Euler PDE versus Wave PDE:
When is the Euler differential equation hyperbolic?
When is the Euler differential equation qualitatively similar to the wave equation?
When is it qualitatively different?

Review: Elliptic/ Parabolic/ Hyperbolic

d'Alembert's solution
What is the general solution of the wave equation

$$U_{tt} = c^2 U_{xx}$$

in terms of two arbitrary functions?

Laplace's solution.
What is the general solution of Laplace's equation

$$U_{xx} + U_{yy} = 0$$

in terms of two arbitrary functions?

Other standard Euler PDEs

Additional examples of PDEs of the generalized constant-coefficient Euler class are:

- Klein–Gordon equation:

$$\partial_t^2 \phi - \nabla^2 \phi = -m^2 \phi$$

  - This generalizes the wave equation.
  - In particle physics, suitable for a scalar particle with mass. (For example, the Higgs particle after spontaneous symmetry breaking. Keep your eye on the LHC in France/ Switzerland for details...)

Other standard Euler PDEs

- Klein–Gordon equation:
  - Also used in plasma physics, where it is useful for describing screening effects. ($m \longleftrightarrow$ Debye screening length.)
  - Also used in super-conductivity — m is then related to the London flux penetration depth.
  - Also useful for a string in a valley.
- In terms of the generalized Euler PDE

$$a\; U_{xx} + 2h\; U_{xy} + b\; U_{yy} + c\; U_x + d\; U_y + e\; U + f = 0$$

  the Klein–Gordon equation corresponds to

$$a \to 1; \quad h \to 0; \quad b \to -1;$$

$$c \to 0; \quad d \to 0; \quad e \to m^2; \quad f \to 0$$

  with the notational change $x \to t$, $y \to x$.
- There is a natural generalization from (1+1) to (2+1) and (3+1) dimensions.

Other standard Euler PDEs

- Helmholtz equation:

$$\nabla^2 \phi = m^2 \phi.$$

  - Generalizes Laplace's equation.
  - Often results from the wave equation after "separation of variables" — lots more on this later!
  - Also used in early nuclear physics — the pion potential:

$$\phi = -\frac{\exp(-mr)}{r}; \qquad F = -\nabla\phi = -\exp(-mr)\,\frac{1+mr}{r^2}.$$

  - Note modification of "inverse square" law.

Other standard Euler PDEs

- In terms of the generalized Euler PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} + c\ U_x + d\ U_y + e\ U + f = 0$$

the Helmholtz equation corresponds to

$$a \to 1; \quad h \to 0; \quad b \to 1;$$

$$c \to 0; \quad d \to 0; \quad e \to m^2; \quad f \to 0.$$

- There is a natural generalization to three space dimensions.

Other standard Euler PDEs

- Maxwell equations (source free):

$$\nabla \cdot E = 0$$
$$curl B - \partial_t E = 0$$
$$\nabla \cdot B = 0$$
$$curl E + \partial_t B = 0$$

- These PDEs link the space and time dependence of electric and magnetic fields.
- (Thankfully they are linear PDEs, which is why we can do such a lot with them.)
- These equations are very well understood and underly much of humanity's pre-quantum technology.

Other standard Euler PDEs

- The Maxwell equations can be put into the form of a system of Euler PDEs, with electric fields coupled to magnetic fields.
- For a small challenge, use the rules of vector calculus to derive wave equations for E and B:

$$\partial_t^2 E - \nabla^2 E = 0$$

$$\partial_t^2 B - \nabla^2 B = 0$$

- Note that for simplicity I have adopted units where the speed of light equals unity.

Summary

By now I hope you are convinced of the central importance of the Euler PDE, both in its original form and in the generalized constant-coefficient case.

(And later on we'll see even more generalizations.)

End: