Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

Matt Visser

21 February 2024

Outline:

Administrivia

Administrivia:

- Lectures:
    - Monday; 12:00–12:50; MYLT 102.
    - Tuesday; 12:00–12:50; MYLT 220.
    - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
    - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
    - Part 1: Matt Visser.
    - Part 2: Dimitrios Mitsotakis.

d’Alembert’s solution to the wave equation

General solution and boundary conditions:
Suppose $U(x,t)$ satisfies the wave equation

$$\boxed{U_{xx} - \frac{1}{c^2}\, U_{tt} = 0}$$

and suppose that the BC (initial conditions) are:

$$U(x,0) = f(x),$$

$$U_t(x,0) = g(x).$$

For example, $U(x,t)$ could be the displacement of an infinitely long stretched string, set vibrating from its equilibrium position along the x-axis by starting it off with the shape defined by $f(x)$ and the velocity $g(x)$.

d'Alembert's solution to the wave equation:

The general solution to this equation is

$$U(x,t) = F(x+ct) + G(x-ct)$$

where F and G are arbitrary functions.

- But we do not at this stage know what F and G look like in terms of our "given" data, f and g; that is the problem we will now solve.

Applying the conditions, we have

$$F(x) + G(x) = f(x)$$

$$c\,F'(x) - c\,G'(x) = g(x)$$

where $'$ denotes derivative.

These can be solved to find

$$F(x) = \frac{1}{2c}\int_a^x g(s)\,\mathrm{d}s + \frac{1}{2}f(x) \qquad \text{(here } a \text{ is arbitrary)}$$

and

$$G(x) = -\frac{1}{2c}\int_a^x g(s)\,\mathrm{d}s + \frac{1}{2}f(x)$$

so that the general solution, presented in terms of the initial data f and g, is simply:

$$\boxed{U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s}$$

d'Alembert's solution to the wave equation:

If, for example, the string was released from rest, so that $g(x) = 0$, and if its initial displacement is $f(x)$, we find

$$\boxed{U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$$

which shows that (half of) the displacement travels down the string in each direction, keeping its shape, with velocity c.

Thus we can interpret the constant c in the wave equation as the velocity of the ensuing waves of vibration.

Question

How does this generalize to more than one space dimension?
In fact, does this generalize to more than one space dimension?

Difficulties with d'Alembert's solution:

WARNING: Many problems have equations for which the general solution is easy to find, but for which other conditions, (boundary conditions, initial conditions), make it impossible to find an explicit solution by using this general solution.

In d'Alembert's solution, for example, we knew the initial shape of the string, so we required our general solution to also satisfy the Boundary Conditions, leading to the functional equations:

$$F(x) + G(x) = f(x)$$

$$c\ F'(x) - c\ G'(x) = g(x)$$

for the arbitrary functions F and G.

In d'Alembert's case, these were easy equations to solve...

Difficulties with d'Alembert's solution:

In d'Alembert's case, these were easy equations to solve...

But in many other cases, the functional equations that result from the BC are extremely difficult to solve.

(This is one of the reasons why we are not too concerned if it's not possible to write down a "general solution" for a given PDE.
Though nice to have available, once boundary conditions are imposed "general solutions" are not always as useful as one might hope.)

Consult the second heat equation example of the of the SOV problems/ exercises below.

There, the general solution is obvious, but the equations resulting from the boundary conditions are practically impossible to solve.

Exercise:

Show that

$$U(x, y, t) = F(x + iay - vt) + G(x - iay - vt)$$

where F and G are arbitrary twice differentiable functions, is a general solution (in the sense we have defined it) of the (2+1) dimensional wave equation

$$U_{xx} + U_{yy} = \frac{1}{c^2}\, U_{tt}$$

when

$$a^2 = 1 - v^2/c^2.$$

Generalize...

Applying d'Alembert's solution:

Exercise:

Solve the wave equation $U_{xx} - U_{tt} = 0$, given that $U(x,0) = B(x)$, and $U_t(x,0) = 0$, where $B(x)$ is the bump function

$$B(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the shape of $U(x,t)$ at some future times $t > 0$; say $t = 2$, 4, 6, and 8.

What is the wave velocity?

End: