Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Spring 2024

Matt Visser

21 February 2024

Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Fourier Series

Fourier series:

Based on the example we used to describe the SOV principle, we found strong reasons for suspecting that relatively general functions $f(x)$ should be representable as sums of sines and cosines:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \left[A_n \cos(\pi n x/L) + B_n \sin(\pi n x/L)\right]}$$

In this chapter we will ask (and answer) how general this sort of decomposition is, and how to calculate the coefficients A_n and B_n.

Fourier coefficients:

As it turns out, calculating the coefficients is easy:

Suppose we have a function $f(x)$ defined on the interval $(0, L)$, and suppose that in that interval it is described by a Fourier series

$$f(x) = \sum_{n=0}^{\infty} [A_n \; \cos(\pi n x/L) + B_n \; \sin(\pi n x/L)]$$

which we shall (for now) simply assume converges, (at least "almost everywhere", in some point-wise sense).

Fourier coefficients:

Warning

There is nothing sacred about the use of the interval $[0, L]$.
Any interval $[a, b]$ could be used as long as you are willing to translate and rescale the domain of the function.
You could, for instance always choose to work on the domain $[0, 1]$.
Working on $[0, L]$ is a compromise between complete generality and obtaining tractable equations.

Note that the Fourier sum is automatically periodic under $x \to x + 2L$, even if the original function $f(x)$ is undefined outside of this range.

$$f(x) = \sum_{n=0}^{\infty} [A_n \cos(\pi n x / L) + B_n \sin(\pi n x / L)]$$

Fourier coefficients:

Now consider the four integrals:

$$\int_{-L}^{+L} \cos(\pi n x/L)\ \cos(\pi m x/L)\, \mathrm{d}x = L\,(\delta_{mn} + \delta_{m0}\ \delta_{n0})$$

$$\int_{-L}^{+L} \sin(\pi n x/L)\ \sin(\pi m x/L)\, \mathrm{d}x = L\,(\delta_{mn} - \delta_{m0}\ \delta_{n0})$$

$$\int_{-L}^{+L} \sin(\pi n x/L)\ \cos(\pi m x/L)\, \mathrm{d}x = 0$$

$$\int_{-L}^{+L} \cos(\pi n x/L)\ \sin(\pi m x/L)\, \mathrm{d}x = 0$$

$$\delta_{mn} = \begin{cases} 1 \text{ if } m = n; \\ 0 \text{ if } m \neq n. \end{cases}$$

Fourier coefficients:

Proof: Part 1 of 4.

For example, suppose to start with that both $n+m$ and $n-m$ are nonzero. Then, taking $x = L\,z$, so $dx = L\,dz$, we have

$$\begin{aligned}\int_{-L}^{+L} \cos(\pi n x/L)\ \cos(\pi m x/L)\,\mathrm{d}x &= L\int_{-1}^{+1} \cos(\pi n z)\ \cos(\pi m z)\,\mathrm{d}z \\ &= \frac{L}{2}\ \int_{-1}^{+1} \{\cos(\pi[n+m]z) + \cos(\pi[n-m]z)\}\ \mathrm{d}z \\ &= \frac{L}{2}\ \frac{1}{\pi}\left\{\frac{1}{n+m}\sin(\pi[n+m]z)|_{-1}^{+1} + \frac{1}{n-m}\sin(\pi[n-m]z)|_{-1}^{+1}\right\} \\ &= 0.\end{aligned}$$

Thus this integral is definitely zero if both $n+m$ and $n-m$ are nonzero. $\square$

Fourier coefficients:

Proof: Part 2 of 4.

If $n+m=0$ but $n-m\neq 0$, (i.e., $n=-m\neq 0$), then

$$\int_{-L}^{+L}\cos(\pi nx/L)\,\cos(\pi mx/L)\,\mathrm{d}x = L\int_{-1}^{+1}\cos(\pi nz)\,\cos(\pi mz)\,\mathrm{d}z$$
$$= L\int_{-1}^{+1}\cos^2(\pi nz)\,\mathrm{d}z$$
$$= L\times 2\times\frac{1}{2}$$
$$= L$$

$\square$

Proof: Part 3 of 4.

Similarly if $n - m = 0$ but $n + m \neq 0$, (i.e., $n = m \neq 0$), then

$$\begin{aligned}
\int_{-L}^{+L} \cos(\pi n x/L)\;\cos(\pi m x/L)\,\mathrm{d}x &= L\int_{-1}^{+1} \cos(\pi n z)\;\cos(\pi m z)\,\mathrm{d}z \\
&= L\;\int_{-1}^{+1} \cos^2(\pi n z)\,\mathrm{d}z \\
&= L \times 2 \times \frac{1}{2} \\
&= L
\end{aligned}$$

$\square$

Proof: Part 4 of 4.

Finally if $n = m = 0$

$$\int_{-L}^{+L} \cos(\pi n x/L)\cos(\pi m x/L)\,\mathrm{d}x = \int_{-L}^{+L} 1\,\mathrm{d}x$$

$$= 2L$$

$\square$

Fourier coefficients:

Proof.

Collecting these results

$$\int_{-L}^{+L} \cos(\pi n x/L)\ \cos(\pi m x/L)\,\mathrm{d}x = L\ (\delta_{mn} + \delta_{m0}\ \delta_{n0})$$

The other three integrals are just minor variations on this theme. □

Exercise

Check the other three integrals.
In particular, we easily see

$$\int_{-L}^{+L} \sin(\pi n x/L)\ \sin(\pi m x/L)\,\mathrm{d}x = L\ (\delta_{mn} - \delta_{m0}\ \delta_{n0})$$

The last two integrals are trivial.

Fourier coefficients:

Proof: sin − sin integral.

For example, suppose to start with that both $n+m$ and $n-m$ are nonzero. Then

$$\begin{aligned}
\int_{-L}^{+L} \sin(\pi n x/L)\ \sin(\pi m x/L)\,\mathrm{d}x &= L\int_{-1}^{+1} \sin(\pi n z)\ \sin(\pi m z)\,\mathrm{d}z \\
&= \frac{L}{2}\ \int_{-1}^{+1} \{-\cos(\pi[n+m]z) + \cos(\pi[n-m]z)\}\ \mathrm{d}z \\
&= \frac{L}{2}\ \frac{2}{\pi} \left\{ -\frac{1}{n+m} \sin(\pi[n+m]z)|_{-1}^{+1} + \frac{1}{n-m} \sin(\pi[n-m]z)|_{-1}^{+1} \right\} \\
&= 0.
\end{aligned}$$

Thus this integral is definitely zero if both $n+m$ and $n-m$ are nonzero. Other sub-cases obvious... $\square$

Fourier coefficients:

Proof: sin – cos and cos – sin integrals.

$$\int_{-L}^{+L} \sin(\pi n x/L)\;\cos(\pi m x/L)\,\mathrm{d}x = 0$$

$$\int_{-L}^{+L} \cos(\pi n x/L)\;\sin(\pi m x/L)\,\mathrm{d}x = 0$$

Why are these two integrals obvious? □

Fourier coefficients:

So now we play a trick.
Take $f(x)$ to be defined in $x \in (0, L)$ and extend it, in an arbitrary way, to a function $\hat{f}(x)$ defined on $x \in [-L, +L]$.

Warning

There is again nothing sacred about the use of the interval $[-L, L]$.

Any interval $[a, b]$ could be used, as long as you are willing to translate and rescale the domain of the function.

You could, for instance, always choose to work on the domain $[-1, 1]$.

Working on $[-L, L]$ is a compromise between complete generality and obtaining tractable equations.

Fourier coefficients:

Assume that $\hat{f}(x)$, defined on $[-L, L]$, possesses a Fourier series

$$\hat{f}(x) = \sum_{n=0}^{\infty} [A_n \cos(\pi n x/L) + B_n \sin(\pi n x/L)]$$

Now multiply both sides of this equation by $\cos(\pi m x/L)$ and integrate from $-L$ to $+L$.

$$\int_{-L}^{+L} \cos(\pi m x/L)\, \hat{f}(x)\, \mathrm{d}x = \sum_{n=0}^{\infty} [A_n\ L\ (\delta_{mn} + \delta_{m0}\ \delta_{n0})]$$

Fourier coefficients:

Then the sum over n is easily done and

$$\boxed{\begin{gathered} A_0 = \frac{1}{2L} \int_{-L}^{+L} \hat{f}(x)\, \mathrm{d}x. \\[2em] A_{n\neq 0} = \frac{1}{L} \int_{-L}^{+L} \cos(\pi n x/L)\; \hat{f}(x)\, \mathrm{d}x. \end{gathered}}$$

Fourier coefficients:

Similarly, if we multiply both sides of this equation by $\sin(\pi m x/L)$, and integrate from $-L$ to $+L$ we have

$$\int_{-L}^{+L} \sin(\pi m x/L)\, \hat{f}(x)\, \mathrm{d}x = \sum_{n=0}^{\infty} [B_n\, L\, (\delta_{mn} - \delta_{m0}\, \delta_{n0})]$$

Then, summing over n, we have

$$\boxed{\begin{gathered} B_0 = 0. \\[2em] B_{n\neq 0} = \frac{1}{L} \int_{-L}^{+L} \sin(\pi n x/L)\, \hat{f}(x)\, \mathrm{d}x. \end{gathered}}$$

Fourier coefficients:

Currently, these formulae have been derived under the assumption that the series converges.

That is, so far, we simply assume that

$$\boxed{\hat{f}(x) = \sum_{n=0}^{\infty} \left[A_n \cos(\pi n x/L) + B_n \sin(\pi n x/L)\right],}$$

with $\hat{f}(x)$ defined on $[-L, +L]$, makes sense!

Fourier coefficients:

Summary:

$$\hat{f}(x) = \sum_{n=0}^{\infty} \left[A_n \cos(\pi n x/L) + B_n \sin(\pi n x/L)\right],$$

$$A_0 = \frac{1}{2L} \int_{-L}^{+L} \hat{f}(x)\, \mathrm{d}x.$$

$$A_{n\neq 0} = \frac{1}{L} \int_{-L}^{+L} \cos(\pi n x/L)\; \hat{f}(x)\, \mathrm{d}x.$$

$$B_0 = 0.$$

$$B_{n\neq 0} = \frac{1}{L} \int_{-L}^{+L} \sin(\pi n x/L)\; \hat{f}(x)\, \mathrm{d}x.$$

Fourier coefficients:

Remarks:

- These formulae for the coefficients are called the Euler–Fourier formulae.
- (Or sometimes just the Euler formulae — Euler did a tremendous amount of research on PDEs.)
- The above shows how to find A_m and B_m given that $f(x)$ is extended in some arbitrary way to $\hat{f}(x)$, and given that $\hat{f}(x)$ can be written as an infinite Fourier series.
- It does not (yet) follow that, if you were to calculate the A_m and B_m by this prescription, and put these values of A_m and B_m back into the series, that the resulting series would always converge to $f(x)$.
- (In fact it does not always converge; at best it is convergent "almost everywhere".)

Fourier coefficients:

- There is a large degree of arbitrariness in the prescription — $f(x)$ can be extended to $\hat{f}(x)$ in an arbitrary way and we still seem to get a sensible Fourier series?
- What on earth is going on here?
- [Explanation below.]
- A necessary condition for the Fourier series to exist is that the Fourier coefficients be well defined, which in turn requires (at the very least), that $\hat{f}(x)$ be integrable.
- Eg, remember various first year courses: at least for a finite number of finite discontinuities in $\hat{f}(x)$ the existence of the A_n and B_n is safe...
- Now let's try for some sufficient conditions.

Definition (Piecewise Continuity)

A function $f(x)$ is piecewise continuous on the interval $a < x < b$ if the interval can be partitioned into a finite number of sub-intervals by using the points

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

in such a way that:

- $f(x)$ is continuous on each of the open subintervals (x_i, x_{i+1}).
- $f(x)$ approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

That is, if

$$f(x_i^+) = \lim_{h \to 0+} f(x_i + h) \qquad \text{and} \qquad f(x_i^-) = \lim_{h \to 0-} f(x_i + h)$$

both exist and are finite for all $i = 0, 1, 2, \ldots n$.

Catchphrase: "At worst a finite number of finite discontinuities..."

Fourier coefficients:

Theorem (Fourier's general theorem:)

Suppose that the functions $\hat{f}(x)$ and $\hat{f}'(x)$ are both piecewise continuous on the interval $-L \leq 0 \leq L$, then:

- *$\hat{f}(x)$ has a Fourier series whose coefficients are determined by the Euler–Fourier formulae above.*
- *The Fourier series converges to $\hat{f}(x)$ at all points where $\hat{f}(x)$ is continuous.*
- *The Fourier series converges to $\frac{1}{2}[\hat{f}(x^+) + \hat{f}(x^-)]$ at points of discontinuity.*

Remarks:

- The conditions of this theorem are certainly sufficient for the convergence of the Fourier series.
- They are not necessary.
- Further, they are not even the most general sufficient conditions.
- As far as I can tell, nobody knows the (minimal) necessary and sufficient conditions for the Fourier series to converge to the function almost everywhere.
- That is, we know some necessary conditions, and we know some sufficient conditions, but as far as I can tell no-one knows the (minimal) necessary and sufficient conditions for convergence.

Fourier coefficients:

Proof.

Convergence of the Fourier series:

We here reproduce Kreyszig's (simplified) proof of convergence for the Fourier series for (particularly simple) functions $\hat{f}(x)$ which are continuous, have continuous second derivatives, and which are periodic with period $2L$.

This convergence theorem is useful because of its simplicity, and because it illustrates the use of convergence theorems you should already have seen. $\square$

Fourier coefficients:

Proof (continued).

The more general case enunciated above, (for piecewise continuous functions), and the proof that it actually converges to the values stated, requires more analysis than we have done.

Note that under the simplified conditions of Kreyszig's simplified theorem,

$$\hat{f}(-L) = \hat{f}(L) \qquad \text{and} \qquad \hat{f}'(-L) = \hat{f}'(L).$$

$\square$

Fourier coefficients:

Proof (continued).

Integrating the Euler–Fourier formulae (for $n \neq 0$) by parts we find that

$$\begin{aligned} A_n &= \frac{1}{L} \int_{-L}^{+L} \cos(\pi n x/L)\, \hat{f}(x)\, \mathrm{d}x \\ &= \left. \frac{\hat{f}(x)\ \sin(\pi n x/L)}{n\pi} \right|_{-L}^{+L} - \frac{1}{n\pi} \int_{-L}^{+L} \sin(\pi n x/L)\, \hat{f}'(x)\, \mathrm{d}x \\ &= -\frac{1}{n\pi} \int_{-L}^{+L} \sin(\pi n x/L)\, \hat{f}'(x)\, \mathrm{d}x \end{aligned}$$

(The contributions from upper and lower limits vanish because the sine function is zero there.)

$\square$

Fourier coefficients:

Proof (continued).

Now integrate by parts a second time

$$
\begin{aligned}
A_n &= \left.\frac{\hat{f}'(x)\ \cos(\pi n x/L)}{n\pi\ (n\pi/L)}\right|_{-L}^{+L} - \frac{1}{n\pi\ (n\pi/L)} \int_{-L}^{+L} \cos(\pi n x/L)\ \hat{f}''(x)\ \mathrm{d}x \\
&= -\frac{L}{n^2\pi^2} \int_{-L}^{+L} \cos(\pi n x/L)\ \hat{f}''(x)\ \mathrm{d}x
\end{aligned}
$$

(The contributions from upper and lower limits now cancel because the derivative is assumed to be periodic.)

$\square$

Fourier coefficients:

Proof (continued).

But now, because $\hat{f}(x)$ by assumption has a continuous second derivative on $[-L, +L]$, it must be bounded

$$|\hat{f}''(x)| < M$$

Therefore

$$|A_n| < \frac{L}{n^2\pi^2} \int_{-L}^{L} |\cos(\pi n x/L)\; \hat{f}''(x)|\, \mathrm{d}x < \frac{L}{n^2\pi^2} \int_{-L}^{L} M\, \mathrm{d}x < \frac{2ML^2}{n^2\pi^2}$$

$\square$

Fourier coefficients:

Proof (continued).

Similarly, we can bound the B_n for all n (just repeat the analogous steps)

$$|B_n| < \frac{2ML^2}{n^2\pi^2}$$

But then

$$|\text{Fourier series}| < |A_0| + \frac{4ML^2}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\dots\right)$$

And this series definitely does converge.
Therefore the Fourier series converges. $\square$

Fourier coefficients:

It is a standard result (the Basel problem, solved by Euler) that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots = \zeta(2) = \frac{\pi^2}{6}$$

- (For the dedicated)
  In fact by the Weierstrass test the Fourier series converges uniformly; which ultimately justifies the way we have cavalierly interchanged summations and integrations.
- (For the dedicated)
  A considerably more subtle proof is needed if you want to get away with piecewise continuity as your only input assumption.
- Note:

$$|\text{Fourier series}| < |A_0| + \frac{2ML^2}{3}$$

Fourier coefficients:

Periodicity:

- Since $\sin(\pi x/L)$ and $\cos(\pi x/L)$ are functions which are periodic with period $2L$, it follows that the Fourier series are themselves functions which are periodic with period $2L$.
- Thus, unless the function $\hat{f}(x)$ has the same period, the Fourier series and the function it is obtained from can only agree on the original interval.
- On the other hand, if $\hat{f}(x)$ has period $2L$ then the series and the function agree (almost) everywhere.

Fourier coefficients:

References:

- Advanced Calculus, pp 321 ff.
- Kreyszig, E. Advanced Engineering Mathematics, pp 581 ff.
- In fact, any text on Engineering Mathematics will probably have a discussion of Fourier series.

Fourier sine series:

Using the freedom of the extension process:

Now we are going to use the freedom of the extension process

$$f : [0, L] \to \hat{f} : [-L, L]$$

to see if we can come up with simpler versions of the Fourier series.

Suppose we construct $\hat{f}(x)$ so that it is odd in the interval $[-L, L]$.

That is:

$$\begin{aligned} \hat{f}(x) &= f(x) && \text{for} \quad x \in (0, L) \\ \hat{f}(x) &= -f(-x) && \text{for} \quad x \in (-L, 0) \end{aligned}$$

Fourier coefficients:

Then in the Euler–Fourier formulae all the coefficients A_n are zero, and so we have

$$\boxed{f(x) = \sum_{n=1}^{\infty} [B_n \sin(\pi n x/L)]}$$

Here

$$\boxed{B_n = \frac{1}{L}\int_{-L}^{+L} \sin(\pi n x/L)\; \hat{f}(x)\, \mathrm{d}x = \frac{2}{L}\int_{0}^{L} \sin(\pi n x/L)\; f(x)\, \mathrm{d}x}$$

But then we can use the general Fourier theorem to obtain the more specific result below:

Fourier coefficients:

Theorem (Fourier sine theorem)

If $f(x)$ is piecewise continuous, with piecewise continuous derivatives, then the Fourier sine series above converges for all values of x in the interval $[0, L]$.
Furthermore:

i. *If x is a point in $(0, L)$ where $f(x)$ is continuous, then the series converges to $f(x)$.*
ii. *If x is a point in $(0, L)$ where f has a discontinuity, then the series converges to*

$$[f(x^+) + f(x^-)]/2.$$

iii. *At the points $x = 0$ and $x = L$, the series converges to $y = 0$. [Not to $f(0)$ and $f(L)$.]*

Fourier coefficients:

Proof.

The proof of the full theorem requires much more analysis than we have developed.

However, there is a proof of convergence given in Kreyszig,
for C^2 functions which are periodic with period $2L$,
which is relatively straightforward.

We have reproduced it above for the full Fourier series case;
and nothing extra is required for the Fourier sine theorem.

$\square$

Using the freedom of the extension process:

As for the sine series:

Suppose we construct $\hat{f}(x)$ so that it is even in the interval $[-L, L]$.

That is:

$$
\begin{aligned}
\hat{f}(x) &= f(x) && \text{for} \quad x \in (0, L) \\
\hat{f}(x) &= +f(-x) && \text{for} \quad x \in (-L, 0)
\end{aligned}
$$

Fourier cosine series:

Then in the Euler–Fourier formulae all the coefficients B_n are zero, and so we have

$$\boxed{f(x) = \sum_{n=0}^{\infty} [A_n \cos(\pi n x/L)]}$$

with

$$\boxed{A_n = \frac{1}{L}\int_{-L}^{L} \cos(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x = \frac{2}{L}\int_{0}^{L} \cos(\pi n x/L)\ f(x)\ \mathrm{d}x}$$

$$\boxed{A_0 = \frac{1}{2L}\int_{-L}^{L} \hat{f}(x)\ \mathrm{d}x = \frac{1}{L}\int_{0}^{L} f(x)\ \mathrm{d}x.}$$

Fourier cosine series:

Theorem (Fourier cosine theorem)

If $f(x)$ is piecewise continuous, with piecewise continuous derivatives, then the Fourier cosine series above converges for all values of x in the interval $[0, L]$.

Furthermore:

i. *If x is a point in $(0, L)$ where $f(x)$ is continuous, then the series converges to $f(x)$.*

ii. *If x is a point in $(0, L)$ where f has a discontinuity, then the series converges to*

$$[f(x+) + f(x-)]/2.$$

iii. *At the points $x = 0$ and $x = L$, the series converges to $f(0)$ and $f(L)$ respectively.*

Fourier cosine series:

Proof.

Again, as for the sine functions. □

Note the full Fourier theorem is applied to $\hat{f}(x)$ in the interval $[-L, L]$; whereas the Fourier cosine theorem tells you about $f(x)$ in the interval $[0, L]$.

Many other results concerning the convergence of Fourier series are known, ranging from the moderately simple result that the series converges at x if $f(x)$ is differentiable at x, to Lennart Carleson's much more sophisticated 1966 result that the Fourier series of an L_2 (square-integrable) function actually converges almost everywhere.

Fourier cosine series:

Important Note:
In the case of the Fourier cosine series, it is common (but not universal) practice to write the series as

$$f(x) = \frac{\bar{A}_0}{2} + \sum_{n=1}^{\infty} \left[\bar{A}_n \cos(\pi n x/L) \right]$$

with

$$\bar{A}_n = \frac{2}{L} \int_0^L \cos(\pi n x/L)\ f(x)\ \mathrm{d}x$$

where the same formula now holds for all $n = 0, 1, 2, 3, \dots$

This has the effect of simplifying the Euler formulae for the coefficients at the cost of putting an explicit 2 in the contribution of the $n = 0$ mode to the Fourier series.

Fourier cosine series:

Personally, if I were to bother doing this at all, I'd go one step further and define

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \, \cos(\pi n x/L),$$

with

$$a_n = \frac{1}{L}\int_0^L \cos(\pi n x/L) \, f(x) \, \mathrm{d}x,$$

so that

$$a_{-n} = a_{+n}.$$

This gets rid of the explicit occurrence of the 2, completely.

There's no explicit 2's anywhere in either the Euler formula or the Fourier series — of course the 2 is now hiding implicitly in the fact that the summation runs from $-\infty$ to $+\infty$.

Fourier coefficients for L_2 (square-integrable) functions:

Lemma

If $\hat{f}(x)$ is an L_2 function then the A_n and B_n exist and are finite.

Proof (somewhat formal):

Define

$$\langle f, g\rangle = \int_{-L}^{+L} f(x)\, g(x)\, \mathrm{d}x$$

This is an "inner product" ("dot product") on function space.
In particular the Cauchy–Schwartz inequality is satisfied

$$|\langle f, g\rangle| \leq \sqrt{\langle f, f\rangle\ \langle g, g\rangle}$$

That is

$$\left|\int_{-L}^{+L} f(x)\, g(x)\, \mathrm{d}x\right| \leq \sqrt{\int_{-L}^{+L} f(x)^2\, \mathrm{d}x \ \int_{-L}^{+L} g(x)^2\, \mathrm{d}x}$$

□

Fourier coefficients for L_2 (square-integrable) functions:

(continued...)

Up to irrelevant factors

$$A_n = \langle \hat{f}, \cos(n\pi x/L) \rangle; \qquad B_n = \langle \hat{f}, \sin(n\pi x/L) \rangle.$$

By the Cauchy–Schwartz inequality

$$|A_n| \leq \sqrt{\langle \hat{f}, \hat{f} \rangle \; \langle \cos(n\pi x/L), \cos(n\pi x/L) \rangle} \; ;$$

$$|B_n| \leq \sqrt{\langle \hat{f}, \hat{f} \rangle \; \langle \sin(n\pi x/L), \sin(n\pi x/L) \rangle} \; .$$

So

$$|A_n| \leq \sqrt{\langle \hat{f}, \hat{f} \rangle}; \qquad |B_n| \leq \sqrt{\langle \hat{f}, \hat{f} \rangle}. \qquad\qquad QED!$$

$\square$

Symmetry:

- Since the $\sin(\pi x/L)$ are odd functions, it follows that the sine series is an odd function.
- Therefore, expressing $f(x)$ as a sine series can only be true for the interval $[0, L]$, unless of course $f(x)$ is itself odd, in which case the sine series agrees with $f(x)$ over the entire interval $[-L, L]$.
- On the other hand, the $\cos(\pi x/L)$ are even functions, so a cosine series is an even function.
- Therefore, expressing $f(x)$ as a cosine series can only be true for the interval $[0, L]$, unless of course $f(x)$ is itself even, in which case the sine series agrees with $f(x)$ over the entire interval $[-L, L]$.

Symmetry:

- If a function $f(x)$ is odd (even) then the full Fourier series for the function has only sine functions (cosine functions) in it.
- Thus we obtain the sine (cosine) series for a function $f(x)$ on $[0, L]$ if we extend $f(x)$ to the interval $[-L, L]$ as an odd (even) function $\hat{f}(x)$ and then take the full Fourier series for it.
- For this reason we really only needed to consider the full Fourier series above!!

Truncation errors:

- Naturally when plotting the Fourier series you will need to truncate.
- As you may surmise from the examples above, the error made in a truncation depends on the point x (for instance, note that near jumps and sharp points in the function the series fluctuates rapidly and the error rises).
- Nevertheless, if you use the orthogonality properties then you can estimate the size of the error.

Examples of Fourier series:

- A Fourier sine series for

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ (2-x) & \text{for } 1 < x < 2 \end{cases}$$

- The coefficients are given by:

$$B_n = \frac{2}{L} \int_0^L \sin(\pi n x/L)\, f(x)\, \mathrm{d}x$$

- Hence, since $f(x)$ is piecewise continuous on $[0,2]$ we can write

$$f(x) = \sum_{n=1}^{\infty} \left[B_n \sin(\pi n x/L) \right]$$

- The RHS will converge when $x = 0$ and $x = 2$ to 0 (which is $f(0)$ or $f(2)$).
- Hence in fact the series converges to $f(x)$ on the whole interval.

Examples of Fourier series:

- The Fourier cosine series for the same function:
- The coefficients are given by:

$$A_{n\neq 0} = \frac{2}{L}\int_0^L \cos(\pi n x/L)\; f(x)\;\mathrm{d}x$$

$$A_0 = \frac{1}{L}\int_0^L \hat{f}(x)\;\mathrm{d}x = 1/2.$$

- Hence, since $f(x)$ is piecewise continuous on $[0,2]$ we can write

$$f(x) = \sum_{n=0}^{\infty} \left[A_n \cos(\pi n x/L)\right]$$

- The RHS will converge when $x = 0$ and $x = 2$ to 0 (which is $f(0)$ or $f(2)$).
- Hence in fact the series in this case converges to $f(x)$ on the whole interval.

Some Fourier work:

- Recall the Fourier theorems above.
- In each of the cases below, find the indicated Fourier series for the given function, and, on the same diagram on which you have sketched the function, sketch the first four partial sums (and so watch the series gradually converge to the function).

a. $f(x) = x^2$ for $0 < x < 1$. Find a sine series.

b. $f(x) = x^2$ for $0 < x < 1$. Find a cosine series.

c. $f(x) = 1$ for $0 < x < 1$; $f(x) = -1$ for $-1 < x < 0$.
Find the full Fourier series.

d. $g(x) = \sin x$ for $0 < x < \pi$. Find a Fourier cosine series.

e. $h(x) = \sin(3\pi x)$ for $0 < x < 1$. Find a Fourier sine series.

- Naturally, Maple will be incredibly helpful for drawing the partial sums, and doing integrals!!

Fourier sine and cosine series:

Consider the function $f(x) = \cos(2x)$ for $x \in (0, \pi)$.

1. Find a Fourier cosine series for $f(x)$.
2. Find a Fourier sine series for $f(x)$.

The remaining questions illustrate how you must use the cunning and brilliance honed over years of struggling through Maths courses to solve the problem.

And your common sense.

Boyce and DiPrima, Chapter 10.5, problem 5.
This illustrates how to deal with the case where the end temperatures are kept fixed, but not at zero degrees.

You should consult the relevant part of Boyce and DiPrima.

Let an aluminium rod of length L be initially at the uniform temperature of 25C.

Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0C while the end $x = L$ is heated to 60C, and that both ends are thereafter maintained at those temperatures.

Heat equation using Fourier series:

a. Find the temperature distribution in the rod at any time t.
   Now assume that $L = 20$ cm.
b. Use only the first term in the series for the temperature $U(x,t)$ to find the approximate temperature at $x = 5$ when $t = 30$ sec, and when $t = 60$ sec.
c. Use the first two terms for the series for $U(x,t)$ to find an approximate value of $U(5,30)$.
   What is the percentage difference between the one- and the two-term approximations?
   Does the third term in the series have any appreciable effect for this value of t?
d. Use the first term in the series for $U(x,t)$ to estimate the time that must elapse before the temperature at $x = 5$ comes within 1% of its steady state value.

Heat equation:

Boyce and DiPrima, chapter 10.5, problem 10.

Another heat bar problem, this time with a mixture of end conditions. Find the steady state temperature in a bar that is insulated at the end $x = 0$ and held constant at the end $x = L$.

Question

What does this mean physically?

Heat equation:

Consider the heat equation

$$\partial_t U(t,x) = \partial_x^2 U(t,x)$$

subject to the boundary conditions

$$U(t,-L) = 0 = U(t,L)$$

$$U(0,x) = f(x) = f(-x)$$

That is, $f(x)$ is an even function of x.

1. Using separation of variables find a series representation for $U(t,x)$ that satisfies the boundary conditions at $\pm L$.
2. Then specify the value of the various coefficients in the series on terms of $f(x)$, the initial data at $t=0$.

Boyce and DiPrima, chapter 10.7, problem 6.
This problem requires you to write Laplace's equation in terms of polar coordinates, and then solve by separation of variables.
Find the solution $u(r, \theta)$ of Laplace's equation in the circular sector $r < a$, $0 \leq \theta \leq \pi$, also satisfying the BC

$$u(r, 0) = 0$$

$$u(r, \pi) = 0 \quad \text{for} \quad 0 < r < a$$

$$u(a, \theta) = f(\theta) \quad \text{for} \quad 0 \leq \theta \leq \pi$$

Assume that u is single-valued and bounded in the given region.
In the problem, take

$$f(\theta) = \sin^2(2\theta).$$

Consider $u(r, \theta)$ to be the equilibrium temperature in the sector, when its radial sides are kept fixed at zero degrees, and the arc is heated according to $f(\theta)$.

Laplace's equation:

Use Fourier series to solve Laplace's equation in the square $0 < x, y < 1$ satisfying the boundary conditions

$$U(0, y) = 0$$

$$U(1, y) = 10$$

$$U(x, 0) = 20$$

$$U(x, 1) = 40x(1 - x) = f(x)$$

corresponding to the case of the equilibrium distribution of temperature in a square of gold with edges kept at temperatures of 0, 10, 20, and $f(x)$ degrees respectively.
You will find problems 3, 4 of chapter 10.7 of Boyce and DiPrima very useful, in that they indicate how to deal with the non-zero temperatures.

End: