Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

Matt Visser

21 February 2024

Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Eigenfunction expansions

Eigenfunction expansions:

- Some brief comments to give you a flavour of what can be done.
- I will not prove any of this, but I will simply assert that Fourier series are a very special case of the sort of things that happen with linear ODEs, (and linear PDEs that are separable).
- Any time you have a linear ODE of the form

$$DU = \lambda\, U,$$

with suitable boundary conditions, the solutions of the eigenvalue problem

$$\{U_\alpha(x);\ \lambda_\alpha\}$$

form a complete basis for a large set of functions defined on the domain of the ODE.

Eigenfunction expansions:

- Generically sums like

$$U(x) = \sum_\alpha A_\alpha \; U_\alpha(x)$$

can be used to construct “all interesting” functions on the domain of the ODE.

Eigenfunction expansions:

- In particular, working with the 2-dimensional Laplacian in polar coordinates leads, (after separation of variables), to Bessel's differential equation, the solutions of which (naturally enough) are Bessel functions.
- But this then suggests that we should be able to write Bessel series of the form

$$f(x) = \sum_{\alpha} A_{\alpha} \, J_m(\lambda_{\alpha} x)$$

- Here m, the index of the Bessel function, is related to the number of dimensions of space, while the eigenvalues α are determined by boundary conditions such as (for example)

$$J_m(\lambda_{\alpha} R) = 0.$$

Eigenfunction expansions:

- Convergence and orthogonality properties for these Bessel series (sometimes called Fourier–Bessel series) can be proved by techniques analogous to those used for the ordinary Fourier series.
- Similar games can then be played with the Laplacian in 3 dimensions, leading to spherical harmonics and spherical Bessel functions.

Eigenfunction expansions:

- Ditto for the Schroedinger equation for the simple harmonic oscillator, which leads to Hermite polynomials.
- Eigenfunction expansions are ubiquitous.
- They underlie much of “special function theory” as the special functions are typically defined in terms of the PDE/ ODE you are trying to solve.
- The most general of the “useful” special functions, at least for these purposes, are the hypergeometric functions...
- More general, but less “useful”, (ie, less well understood), are the Heun functions. Still an area of active research.

End: