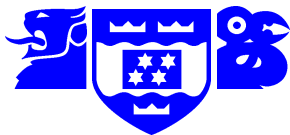


Victoria University of Wellington

*Te Whare Wānanga o te Ūpoko o te Ika a Maui*



— MATH 301 — PDEs —  
Autumn 2024

Matt Visser

21 February 2024



1 Administrivia

2 Eigenfunction expansions

# Administrivia



- **Lectures:**
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- **Tutorial:**
  - Thursday; 12:00–12:50; MYLT 220.
- **Lecturers:**
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.



# Eigenfunction expansions

# Eigenfunction expansions:

- Some brief comments to give you a flavour of what can be done.
- I will not prove any of this, but I will simply assert that Fourier series are a very special case of the sort of things that happen with linear ODEs, (and linear PDEs that are separable).
- Any time you have a linear ODE of the form

$$DU = \lambda U,$$

with suitable boundary conditions, the solutions of the eigenvalue problem

$$\{U_\alpha(x); \lambda_\alpha\}$$

form a complete basis for a large set of functions defined on the domain of the ODE.

# Eigenfunction expansions:

- Generically sums like

$$U(x) = \sum_{\alpha} A_{\alpha} U_{\alpha}(x)$$

can be used to construct “all interesting” functions on the domain of the ODE.

# Eigenfunction expansions:

- In particular, working with the 2-dimensional Laplacian in polar coordinates leads, (after separation of variables), to Bessel's differential equation, the solutions of which (naturally enough) are Bessel functions.
- But this then suggests that we should be able to write Bessel series of the form

$$f(x) = \sum_{\alpha} A_{\alpha} J_m(\lambda_{\alpha} x)$$

- Here  $m$ , the index of the Bessel function, is related to the number of dimensions of space, while the eigenvalues  $\alpha$  are determined by boundary conditions such as (for example)

$$J_m(\lambda_{\alpha} R) = 0.$$



# Eigenfunction expansions:

- Convergence and orthogonality properties for these Bessel series (sometimes called Fourier–Bessel series) can be proved by techniques analogous to those used for the ordinary Fourier series.
- Similar games can then be played with the Laplacian in 3 dimensions, leading to spherical harmonics and spherical Bessel functions.

# Eigenfunction expansions:

- Ditto for the Schrodinger equation for the simple harmonic oscillator, which leads to **Hermite polynomials**.
- Eigenfunction expansions are ubiquitous.
- They underlie much of “special function theory” as the special functions are typically defined in terms of the PDE/ ODE you are trying to solve.
- The most general of the “useful” special functions, at least for these purposes, are the **hypergeometric functions**...
- More general, but less “useful”, (ie, less well understood), are the **Heun functions**. Still an area of active research.



End:

