Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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21 February 2024

Topics:

1. Administrivia
2. General solutions
3. d'Alembert's solution
4. Variable-coefficient Euler PDE
5. Separation of Variables
6. Fourier series

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

General solutions

General solutions:

Two absolutely essential general solutions to memorize:

- Laplace's equation:

$$\boxed{U_{xx} + U_{yy} = 0.}$$

  General solution:

$$\boxed{U(x, y) = F(x + iy) + G(x - iy).}$$

- Wave equation:

$$\boxed{c^2\, U_{xx} - U_{tt} = 0.}$$

  General solution:

$$\boxed{U(x, t) = F(x + ct) + G(x - ct).}$$

Related issues: the Euler PDE...

d’Alembert’s solution

d'Alembert's solution to the wave equation:

- Wave equation:

$$c^2\, U_{xx} - U_{tt} = 0.$$

- Boundary conditions:

$$\forall x \qquad U(x,0) = f(x),$$

$$\forall x \qquad U_t(x,0) = g(x).$$

- General solution:

$$U(x,t) = F(x+ct) + G(x-ct).$$

- Specific solution:

$$U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

Variable-coefficient Euler PDE

Variable-coefficient Euler PDE:

Definition

The generalized (2-dimensional) variable-coefficient Euler PDE is

$$a(x,y)\; U_{xx} + 2h(x,y)\; U_{xy} + b(x,y)\; U_{yy} + c(x,y)\; U_x + d(x,y)\; U_y$$
$$+e(x,y)\; U + f(x,y) = 0,$$

where a, b, h, and c, d, e, f are functions of x and y.

(And at least one of the second-order coefficients $a(x,y)$, $b(x,y)$, or $h(x,y)$, is not identically zero.)

Variable-coefficient Euler PDE:

Definition

The generalized (2-dimensional) variable-coefficient Euler PDE (with non-linear source) is

$$a(x,y)\; U_{xx} + 2h(x,y)\; U_{xy} + b(x,y)\; U_{yy} = F(x,y,U,U_x,U_y),$$

where a, b, are functions of x and y, and F is a function of its indicated arguments.

(And at least one of the second-order coefficients $a(x,y)$, $b(x,y)$, or $h(x,y)$, is not identically zero.)

Variable-coefficient Euler PDE:

Theorem

In 2 dimensions, as long as the 2^{nd}-order coefficients $a(x,y)$, $h(x,y)$, and $b(x,y)$ are not all zero, then you can always divide the plane into disjoint regions in each of which you can, by change of independent variables, bring the generalized variable-coefficient Euler PDE

$$a(x,y)\; U_{xx} + 2h(x,y)\; U_{xy} + b(x,y)\; U_{yy} = F(x,y,U,U_x,U_y),$$

into the form

$$U_{\bar{x}\bar{x}} + \epsilon\; U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x},\bar{y},U,U_{\bar{x}},U_{\bar{y}}),$$

where $\epsilon = \pm 1$ or 0, and $\tilde{F}$ is a function of its indicated arguments. Furthermore

$$\epsilon = sign\left[a(x,y)\; b(x,y) - h(x,y)^2\right].$$

Variable-coefficient Euler PDE:

Theorem

If you want to consider a two dimensional region where $(ab - h^2)$ changes sign, the trick is to use $(ab - h^2)$ as one of your new coordinates, say $\bar{x}$. Then

$$U_{\bar{x}\bar{x}} + \bar{x}\, U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x}, \bar{y}, U, U_{\bar{x}}, U_{\bar{y}}),$$

which is Tricomi's equation with a nonlinear source.

Thus in two dimensions the second-derivative part of the general variable-coefficient Euler equation has been reduced to a very small number of standard cases:

- — Wave equation (with nonlinear source).
- — Laplace's equation (with nonlinear source).
- — Parabolic equation (with nonlinear source).
- — Tricomi's equation (with nonlinear source).

Separation of variables

Separation of Variables:

Basis functions:

Name	Wave	Heat	Laplace
Equation	$u_{xx} = u_{tt}$	$u_{xx} = u_t$	$u_{xx} + u_{yy} = 0$
Ansatz	$X(x)T(t)$	$X(x)T(t)$	$X(x)Y(y)$
	$X''T = XT''$	$X''T = XT'$	$X''Y + XY'' = 0$
SOV	$\frac{X''}{X} = \frac{T''}{T} = -k^2$	$\frac{X''}{X} = \frac{T'}{T} = -k^2$	$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$
	$X = \cos(kx+\phi)$ $T = \cos(kt+\chi)$	$X = \cos(kx+\phi)$ $T = \exp(-k^2 t)$	$X = \cos(kx+\phi)$ $Y = \cosh(ky+\chi)$

(For generality, k is either pure real or pure imaginary.)

General solutions:

Wave	$u(x,t) = \sum_n A_n \cos(k_n x + \phi_n) \cos(k_n t + \chi_n)$
Heat	$u(x,t) = \sum_n A_n \cos(k_n x + \phi_n) \exp(-k_n^2 t)$
Laplace	$u(x,y) = \sum_n A_n \cos(k_n x + \phi_n) \cosh(k_n t + \chi_n)$

(No boundary conditions have yet been used.)

It is only once you add some of the BCs/ICs that the k_n are determined. For example:

- **Dirichlet conditions in the x direction:**

$$u(0,\bullet) = 0 = u(L,\bullet) \qquad \Rightarrow \qquad X(0) = 0 = X(L)$$

$$\Rightarrow \quad \cos(\phi) = 0 = \cos(k_n L + \phi) \quad \Rightarrow \quad \phi = \frac{\pi}{2}; \quad k_n L = n\pi;$$

$$\Rightarrow \qquad k_n = \frac{n\pi}{L}; \qquad X(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Wave	$u(x,t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left\{A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)\right\}$
Heat	$u(x,t) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right) \exp(-\frac{n^2\pi^2}{L^2} t)$
Laplace	$u(x,y) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left\{A_n \cosh\left(\frac{n\pi y}{L}\right) + B_n \sinh\left(\frac{n\pi y}{L}\right)\right\}$

Remaining BC (some examples):

Wave	$u(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$
Wave	$u_t(x,0) = \sum_n \frac{n\pi B_n}{L} \sin\left(\frac{n\pi x}{L}\right)$
Heat	$u(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$
Laplace	$u(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{L}\right)$
Laplace	$u_y(x,0) = \sum_n \frac{n\pi B_n}{L} \sin\left(\frac{n\pi x}{L}\right)$

Apply Fourier sine series...

(Could also apply 2-point BCs in y direction.)

Separation of Variables:

It is only once you add some of the BCs/ICs that the k_n are determined. For example:

- **Neumann conditions in the x direction:**

$$u_x(0,\bullet) = 0 = u_x(L,\bullet) \qquad \Rightarrow \qquad X'(0) = 0 = X'(L)$$

$$\Rightarrow \quad \sin(\phi) = 0 = \sin(k_n L + \phi) \quad \Rightarrow \quad \phi = 0; \quad k_n L = n\pi;$$

$$\Rightarrow \qquad k_n = \frac{n\pi}{L}; \qquad X(x) = \cos\left(\frac{n\pi x}{L}\right).$$

Wave	$u(x,t) = \sum_n \cos\left(\frac{n\pi x}{L}\right)\left\{A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)\right\}$
Heat	$u(x,t) = \sum_n A_n \cos\left(\frac{n\pi x}{L}\right) \exp(-\frac{n^2\pi^2}{L^2} t)$
Laplace	$u(x,y) = \sum_n \cos\left(\frac{n\pi x}{L}\right)\left\{A_n \cosh\left(\frac{n\pi y}{L}\right) + B_n \sinh\left(\frac{n\pi y}{L}\right)\right\}$

Remaining BC (some examples):

Wave	$u(x,0) = \sum_n A_n \cos\left(\frac{n\pi x}{L}\right)$
Wave	$u_t(x,0) = \sum_n \frac{n\pi B_n}{L} \cos\left(\frac{n\pi x}{L}\right)$
Heat	$u(x,0) = \sum_n A_n \cos\left(\frac{n\pi x}{L}\right)$
Laplace	$u(x,0) = \sum_n A_n \cos\left(\frac{n\pi x}{L}\right)$
Laplace	$u_y(x,0) = \sum_n \frac{n\pi B_n}{L} \cos\left(\frac{n\pi x}{L}\right)$

Apply Fourier cosine series...

(Could also apply 2-point BCs in y direction.)

Separation of Variables:

It is only once you add some of the BCs/ICs that the k_n are determined. For example:

- **Mixed conditions in the x direction:**

$$u_x(0,\bullet) = 0 = u(L,\bullet) \qquad \Rightarrow \qquad X'(0) = 0 = X(L)$$

$$\Rightarrow \quad \sin(\phi) = 0 = \cos(k_n L + \phi) \quad \Rightarrow \quad \phi = 0; \quad k_n L = \left(n + \frac{1}{2}\right)\pi;$$

$$\Rightarrow \qquad k_n = \frac{(2n+1)\pi}{2L}; \qquad X(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

W	$u = \sum_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)\left\{A_n \cos\left(\frac{(2n+1)\pi t}{2L}\right) + B_n \sin\left(\frac{(2n+1)\pi t}{2L}\right)\right\}$
H	$u = \sum_n A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right) \exp(-\frac{(n+\frac{1}{2})^2\pi^2}{L^2} t)$
L	$u = \sum_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)\left\{A_n \cosh\left(\frac{(2n+1)\pi y}{2L}\right) + B_n \sinh\left(\frac{(2n+1)\pi y}{2L}\right)\right\}$

Remaining BC (some examples):

Wave	$u(x,0) = \sum_n A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$
Wave	$u_t(x,0) = \sum_n \frac{(2n+1)\pi B_n}{2L} \cos\left(\frac{(2n+1)\pi x}{2L}\right)$
Heat	$u(x,0) = \sum_n A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$
Laplace	$u(x,0) = \sum_n A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)$
Laplace	$u_y(x,0) = \sum_n \frac{(2n+1)\pi B_n}{2L} \cos\left(\frac{(2n+1)\pi x}{2L}\right)$

Apply Fourier cosine series...

(Could apply 2-point BCs in y direction.)

(Could apply even messier 2-point Robin BCs in y direction.)

Fourier series

Fourier series:

General Fourier series:

$$f(x) = \sum_{n=0}^{\infty} \left[A_n \; \cos(\pi n x/L) + B_n \; \sin(\pi n x/L) \right].$$

$$A_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) \, \mathrm{d}x.$$

$$A_{n>0} = \frac{1}{L} \int_{-L}^{+L} \cos(\pi n x/L) \; f(x) \, \mathrm{d}x.$$

$$B_0 = 0.$$

$$B_{n>0} = \frac{1}{L} \int_{-L}^{+L} \sin(\pi n x/L) \; f(x) \, \mathrm{d}x.$$

Periodic on $[-L, L]$.

Fourier series:

Orthogonality:

$$\int_{-L}^{+L} \cos(\pi n x/L)\ \cos(\pi m x/L)\, \mathrm{d}x = L\,(\delta_{mn} + \delta_{m0}\ \delta_{n0})$$

$$\int_{-L}^{+L} \sin(\pi n x/L)\ \sin(\pi m x/L)\, \mathrm{d}x = L\,(\delta_{mn} - \delta_{m0}\ \delta_{n0})$$

$$\int_{-L}^{+L} \sin(\pi n x/L)\ \cos(\pi m x/L)\, \mathrm{d}x = 0$$

$$\int_{-L}^{+L} \cos(\pi n x/L)\ \sin(\pi m x/L)\, \mathrm{d}x = 0$$

Kronecker delta: $\delta_{mn} = \begin{cases} 1 \text{ if } m = n; \\ 0 \text{ if } m \neq n. \end{cases}$

Fourier series:

Theorem (Fourier's general theorem:)

Suppose that the functions $f(x)$ and $f'(x)$ are both piecewise continuous on the interval $-L \leq 0 \leq L$, then:

- *$f(x)$ has a Fourier series whose coefficients are determined by the Euler–Fourier formulae above.*
- *The Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous.*
- *The Fourier series converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ at points of discontinuity.*

Theorem (Fourier sine theorem)

If $f(x)$ is piecewise continuous, with piecewise continuous derivatives, then the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} [B_n \sin(\pi n x/L)]; \qquad B_n = \frac{2}{L}\int_0^{+L} \sin(\pi n x/L)\, f(x)\, \mathrm{d}x;$$

converges for all values of x in the interval $[0, L]$. Furthermore:

i. *If x is a point in $(0, L)$ where $f(x)$ is continuous, then the series converges to $f(x)$.*
ii. *If x is a point in $(0, L)$ where f has a discontinuity, then the series converges to*
$$\frac{1}{2}[f(x^+) + f(x^-)].$$
iii. *At the points $x = 0$ and $x = L$, the series converges to 0. [Not to $f(0)$ and $f(L)$.]*

Fourier series:

Theorem (Fourier cosine theorem)

If $f(x)$ is piecewise continuous, with piecewise continuous derivatives, then the Fourier cosine series

$$f(x) = \sum_{n=0}^{\infty} [A_n \cos(\pi n x/L)]; \qquad A_{n>0} = \frac{2}{L}\int_0^{+L} \cos(\pi n x/L)\, f(x)\, \mathrm{d}x;$$

$$A_0 = \frac{1}{L}\int_0^{+L} f(x)\, \mathrm{d}x;$$

converges for all values of x in the interval $[0, L]$. Furthermore:

i. *If x is a point in $(0, L)$ where $f(x)$ is continuous, then the series converges to $f(x)$.*
ii. *If x is a point in $(0, L)$ where f has a discontinuity, then the series converges to $\frac{1}{2}[f(x^+) + f(x^-)]$.*
iii. *At $x = 0$ and $x = L$, the series converges to $f(0)$ and $f(L)$.*

Fourier series:

- In lectures we proved (Kreyzig) that if $f(x)$ is C^2 and periodic with period $2L$ then there is a constant K such that

$$|A_n| \leq \frac{K}{n^2}; \qquad |B_n| \leq \frac{K}{n^2}.$$

- In tutorials we proved that if $f(x)$ is C^0 then there is a constant K such that

$$|A_n| \leq K; \qquad |B_n| \leq K.$$

(Yes, we did prove this, think about it.)

- In homework you will (hopefully) have proved that if $f(x)$ is C^k and periodic with period $2L$ then there is a constant K such that

$$|A_n| \leq \frac{K}{n^k}; \qquad |B_n| \leq \frac{K}{n^k}.$$

Fourier series:

- In homework you will (hopefully) have proved that if $P(x)$ is a polynomial in x then the Fourier coefficients

$$A_n(P) \text{ and } B_n(P) \text{ are polynomial in } 1/n.$$

- It is extremely common for the Fourier coefficients A_n and B_n to be ratios of polynomials (rational polynomials) in n.

End: