Notes for Assignment 2
Maths 323 fluids 2014

Continuity Equation (Sec 6-7)

Force Balance (Sec 6-8)

Stream Function (Sec 6-9)

Postglacial Rebound (Sec 6-10)

Angle of Subduction (Sec. 6-11)

Diapirs (Sec 6-12)

Stokes Flow (Sec 6-14)

Continuity Eqn:

For incompressible fluids—conservation of fluid "What goes in must come out"

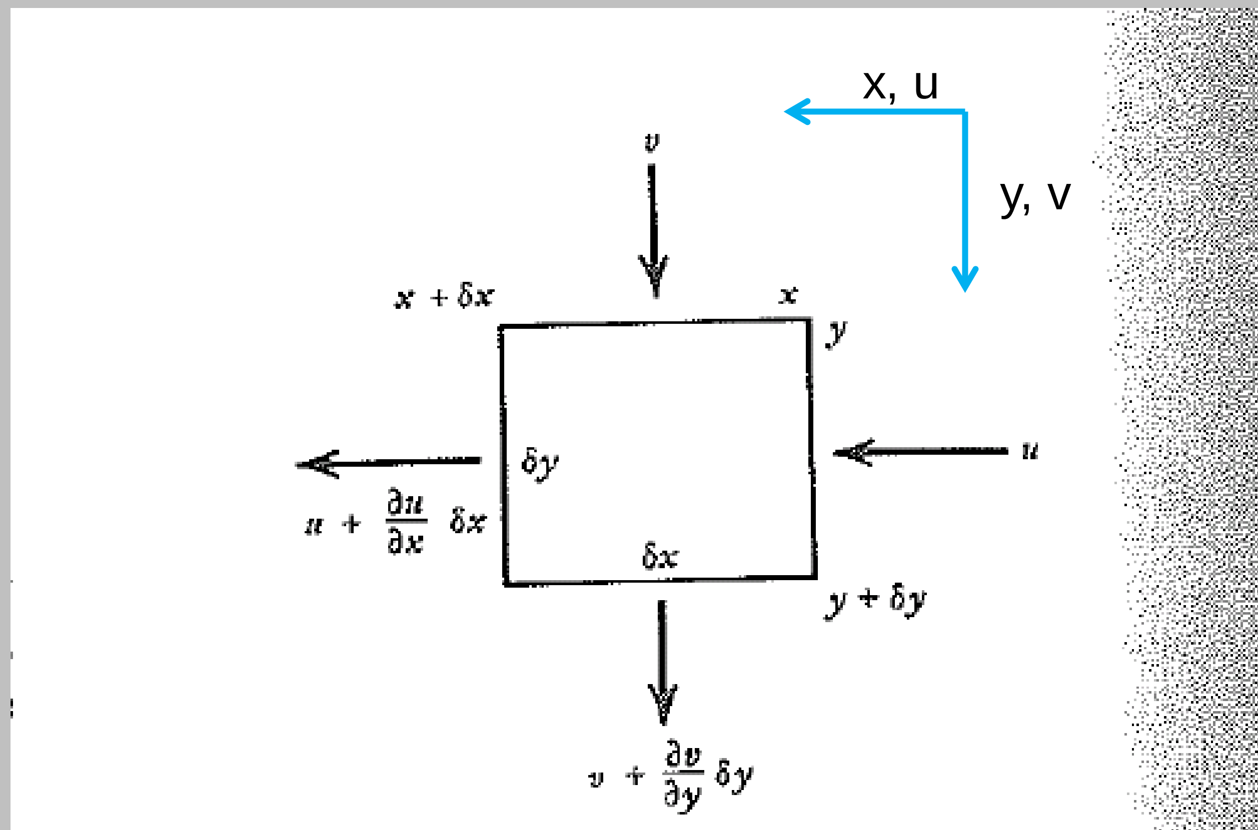
2-D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where

$$u = \frac{\partial x}{\partial t}; v = \frac{\partial y}{\partial t}$$

Note: For 2-D case, often y is used for the vertical direction—for 3-D, usually z is vertical.



6–10 Flow across the surfaces of an infinitesimal rectangular element

Continuity Eqn:

For incompressible fluids—conservation of fluid

2-D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$where$$

$$u = \frac{\partial x}{\partial t}; v = \frac{\partial y}{\partial t}$$

3-D

$$\vec{\nabla} \cdot \vec{u} = 0$$

Or in longer form:

$$\frac{\partial}{\partial x_j}(u_j(\vec{x})) = 0$$

Compressible fluids (3-D)

$$\frac{\partial}{\partial t}(\rho(\vec{x})) + \frac{\partial}{\partial x_j}(\rho(\vec{x})u_j(\vec{x})) = 0$$

Viscous stresses and force balance-2D

$$\vec{F} = m\vec{a}$$

=0 (Neglect acceleration)

Force and acceleration are vectors

$$\vec{F} = \sum_i \vec{f}_i = \sum$$

Pressure forces + Viscous forces + gravity forces

$$\vec{F} = \sum_i \vec{f}_i = (\sum_i (\vec{p}_i a_i + (\sum_j \overleftrightarrow{\tau}_{ij} a_{ij})) + \rho g V \hat{y}$$

a=area,
V=volume
g=acceleration of gravity
p=pressure
τ=stress
ρ=density

Gravity force acts only in vertical (y) direction

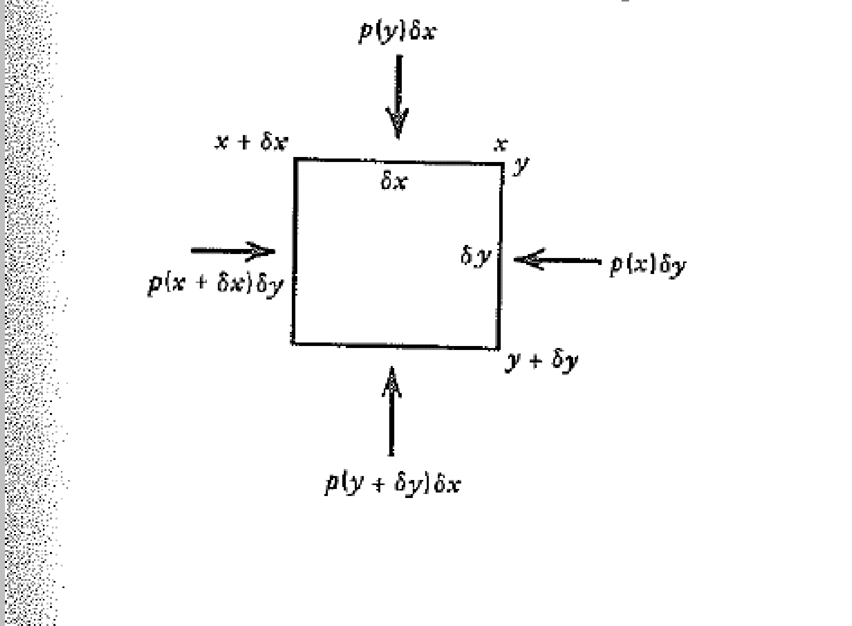
Viscous stresses and force balance-2D

x_1=x
x_2=y
x_3=z

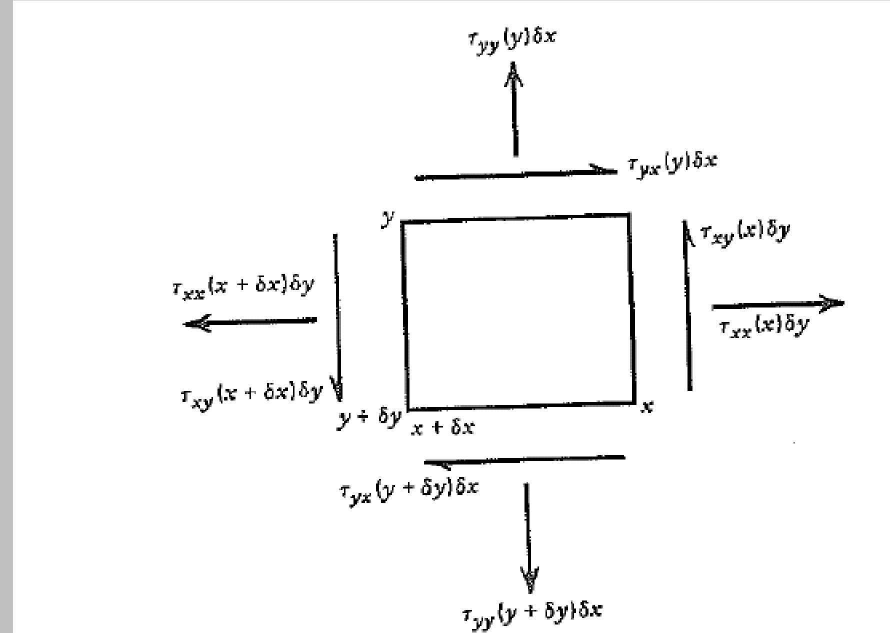
Book uses y=depth (2-D case)

2) Gravity force = ρ*(volume)*g
(just gravity on the element itself)

6–11 Pressure forces acting on an infinitesimal rectangular fluid element.



1) Pressure=pos. inward
—perpendicular to faces-
Often assumed constant or
Given by hydrostatic overburden
(gravity acting on whole column above)



6–12 Viscous forces acting on an infinitesimal two-dimensional rectangular fluid element.

3) Viscous forces are due to
fluid movement and are
parallel or perpendicular to
faces

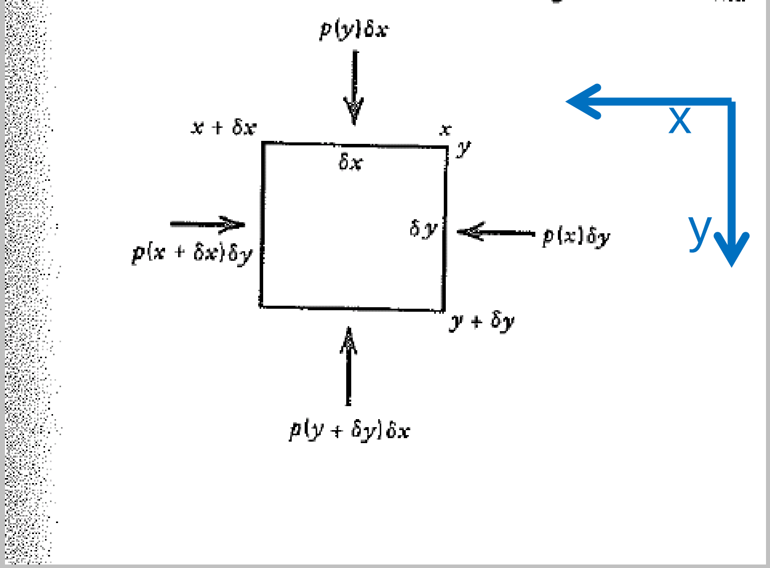
Pressure Forces➔ always inward

$$F=\sum_i f_i=\left(\sum_i (p_i a_i+\left(\sum_j \overleftrightarrow{\tau}_{ij} a_{ij}\right)\right)+\rho g V\hat{y}$$ a=area, V=volume

$$\sum_i p_i a_i=\vec{p}_y(y)dxdz+\vec{p}_x(x)dydz-\vec{p}_y(y+dy)dxdz-\vec{p}_x(x+dx)dydz+(p_z\ldots)$$

Assume 3rd dimension, dz =1; write out explicitly x and y components

$$\sum_i p_i a_i=\vec{p}_y(y)dx+\vec{p}_x(x)dy-\vec{p}_y(y+dy)dx-\vec{p}_x(x+dx)dy$$



6-11 Pressure forces acting on an infinitesimal rectangular fluid element.

Note: Vector nature of pressure—components in both y and x direction (z direction too, but is constant and neglected)

$$\vec{p}_x(x)-\vec{p}_x(x+dx)=\frac{-\partial p}{\partial x}\hat{x}$$

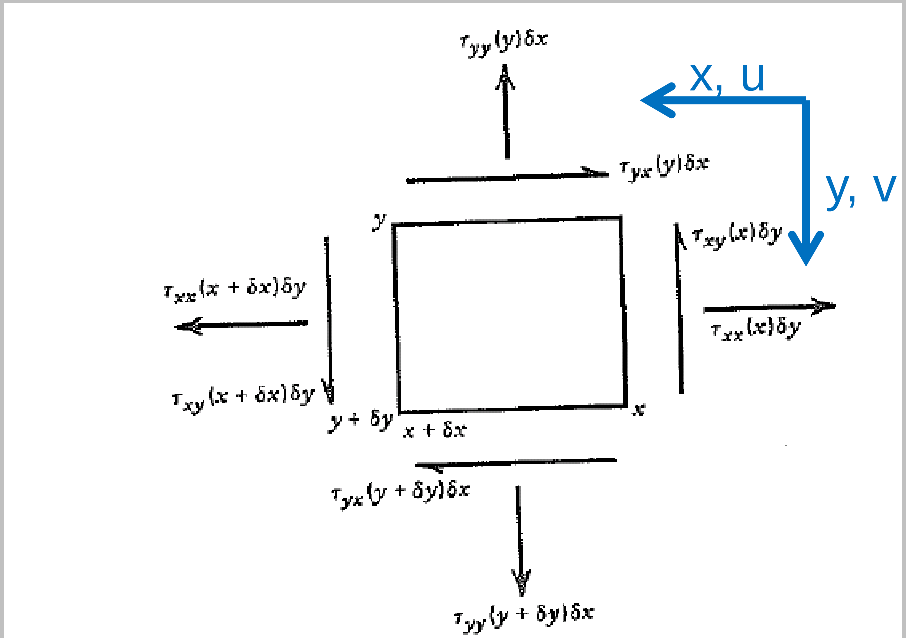
$$\vec{p}_y(y)-\vec{p}_y(y+dy)=\frac{-\partial p}{\partial y}\hat{y}$$

Viscous forces-normal stresses act outwards here

$$F_x(viscous) = \tau_{xx}(x+dx)dydz - \tau_{yx}(y)dxdz - \tau_{xx}(x)dydz + \tau_{yx}(y+dy)dxdz$$

$$F_x(viscous) = \tau_{xx}(x+dx)dy - \tau_{yx}(y)dx - \tau_{xx}(x)dy) + \tau_{yx}(y+dy)dx$$

(Letting dz=1) Similar expression for F_y



6–12 Viscous forces acting on an infinitesimal two-dimensional rectangular fluid element.

Use relationship

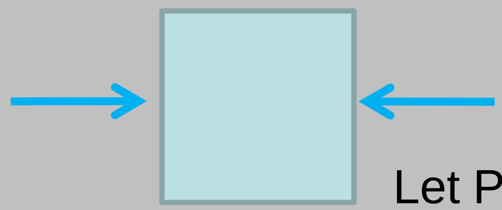
$$\tau_{ij} = \mu\dot{\varepsilon}_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

To get relationship in terms of velocities

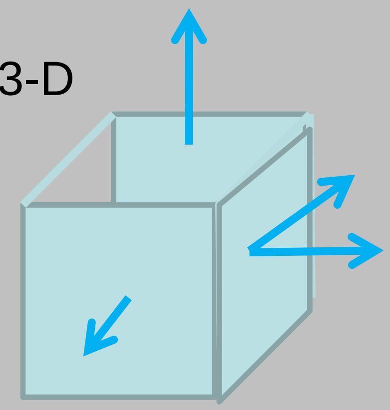
Viscous stresses and force balance

2-D

Pressure=p (pos. inward)



3-D



Let P=p-ρgy=deviatoric stress i.e.
Stress that is different from gravity

τ_{ij}= element of stress tensor, pos. outward

$x_1=x$
$x_2=y$
$x_3=z$

6-56 to 6-58
Book uses y=depth (2-D case)

$$\text{Eq}(6-67)\quad \frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\text{Eq}(6-68)\quad \frac{\partial P}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Force balance, i.e., equation of motion of fluid

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\vec{\tau} = -\mu(\vec{\nabla}\vec{u} + \vec{\nabla}u^T)$$

$$\vec{\nabla}P = \mu\nabla^2\vec{u} \text{ (if no body forces}$$

--3D equivalent to 6-67 and 6-68)

Allowing body forces G_i gives Navier-Stokes

$$\rho G_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_i \partial x_j} = 0$$

Stream Function ψ--a potential

- Like P- and S-wave potentials in seismology, and potentials in quantum mechanics:
- Define ψ such that
- (2-D)

$$u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x}$$

3-D

$$\vec{\psi} = (0,0,\psi)$$

$$\vec{u} = (u,v,0)$$

$$\vec{u} = \vec{\nabla} \times \vec{\psi}$$

$$\text{Eq}(6-67)\ \ \frac{\partial P}{\partial x} = \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$\text{Eq}(6-68)\ \ \frac{\partial P}{\partial y} = \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$$

Substitute into previous equations

Eqn of motion reduces to Biharmonic Eqn:

:

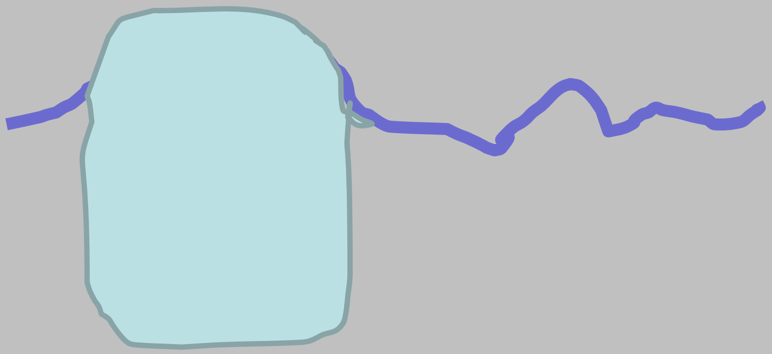
- $\nabla^4 \psi = 0$
- Soln:

$$Q = \int_A^B d\psi = \psi_B - \psi_A$$

- Volumetric flow rate between two points is given by the difference in the stream function

Isostacy

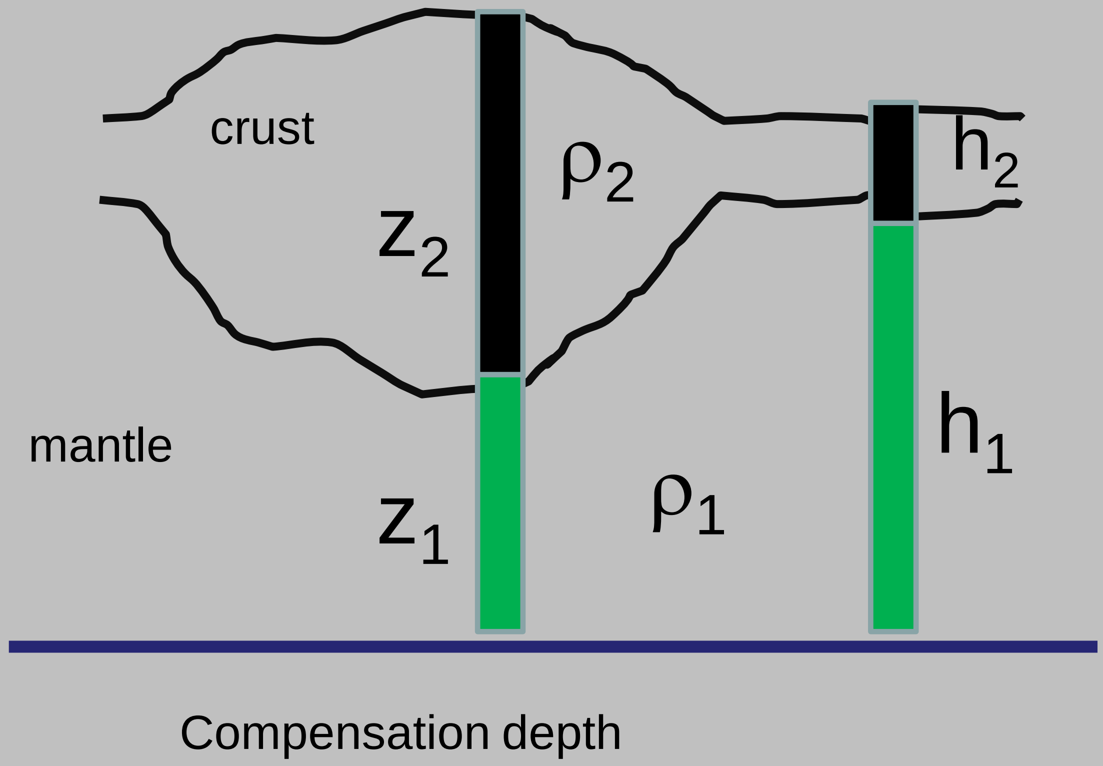
- Solids floating on fluids displace their own weight in the fluid
- E.g., icebergs in water:



- "Airy" Isostacy:

"Airy" Isostacy (constant pressure at a compensation depth)

$\rho_1 h_1 + \rho_2 h_2 = \rho_1 z_1 + \rho_2 z_2$ (total ρgh=constant)



Gravity anomalies

- Earth's gravity field changes due to presence or absence of masses (density differences) of rock/air/water
- This is measurable with very sensitive instruments called gravimeters
- There are several types of corrections that need to be applied to be able to convert the gravity measurements to fields that depend on rock density.

Free air correction

- One of most important is the “free air correction”. It corrects for the height difference between spots on the earth. (Gravity decreases as $1/R^2$ from the center of the Earth so if you are higher, you are further away and gravity is somewhat smaller). Further details are in Turcotte & Schubert Ch 5 or ESCI 305 class.

Free air gravity anomaly

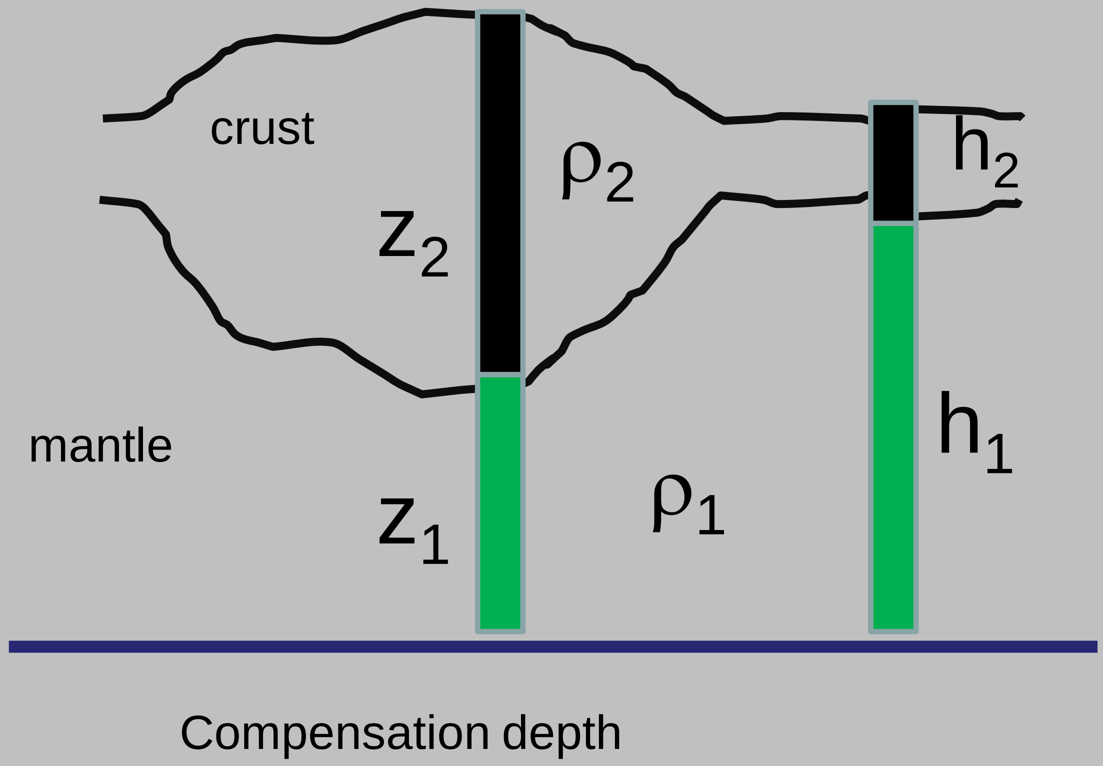
- For the purpose of the assignment question 6-12 all you need to know is that the free air gravity anomaly will be given by

$$g_{FA} = 2\pi\Delta\rho G h$$

- Where $\Delta\rho$=difference in density between two materials (here air vs mantle) G=universal gravitational constant
- h=distance over which density difference occurs

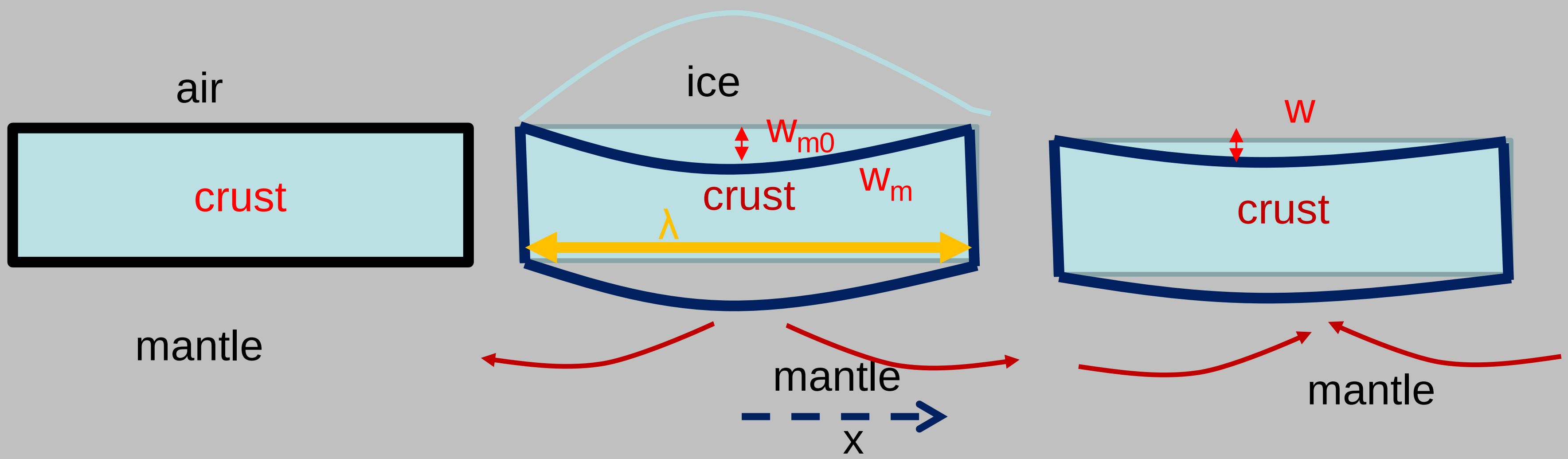
“Airy” Isostacy

$$\rho_1 h_1 + \rho_2 h_2 = \rho_1 z_1 + \rho_2 z_2$$



Glacier effects

- Before glacier during glacier   after glacier



Start just as ice disappears:

$$w_m = w_{m0} \cos\left(\frac{2\pi x}{\lambda}\right)$$

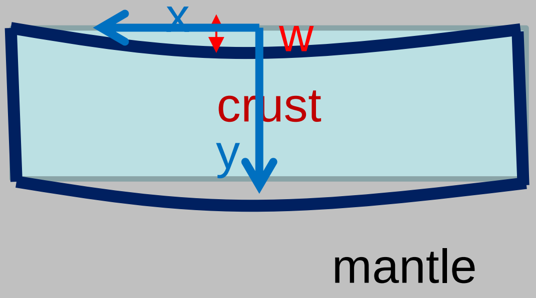
Note that $w_{m0} \ll \lambda$: allows many simplifications

$$w = w_m \exp\left(-\frac{t}{\tau_r}\right)$$

After a time t displacement decreases depending on relaxation time τ_r

Solve biharmonic equation

$$w_m = w_{m0}\cos\left(\frac{2\pi x}{\lambda}\right)$$



- Solve Biharmonic equation $\nabla^4 \psi = 0$

Use Separation of variables:

Assume solution of Eq 6-80

$$\psi = \sin\left(\frac{2\pi x}{\lambda}\right)Y(y)$$

- Show that it works
- Result:6-90 to 6-92
- Surprisingly simple result

$$\psi = A\sin\left(\frac{2\pi x}{\lambda}\right)e^{-2\pi y/\lambda}\left(1+\frac{2\pi y}{\lambda}\right)$$

$$u = \ldots$$

$$y = \ldots$$

$$w = w_m \exp\left(-\frac{t}{\tau_r}\right)$$

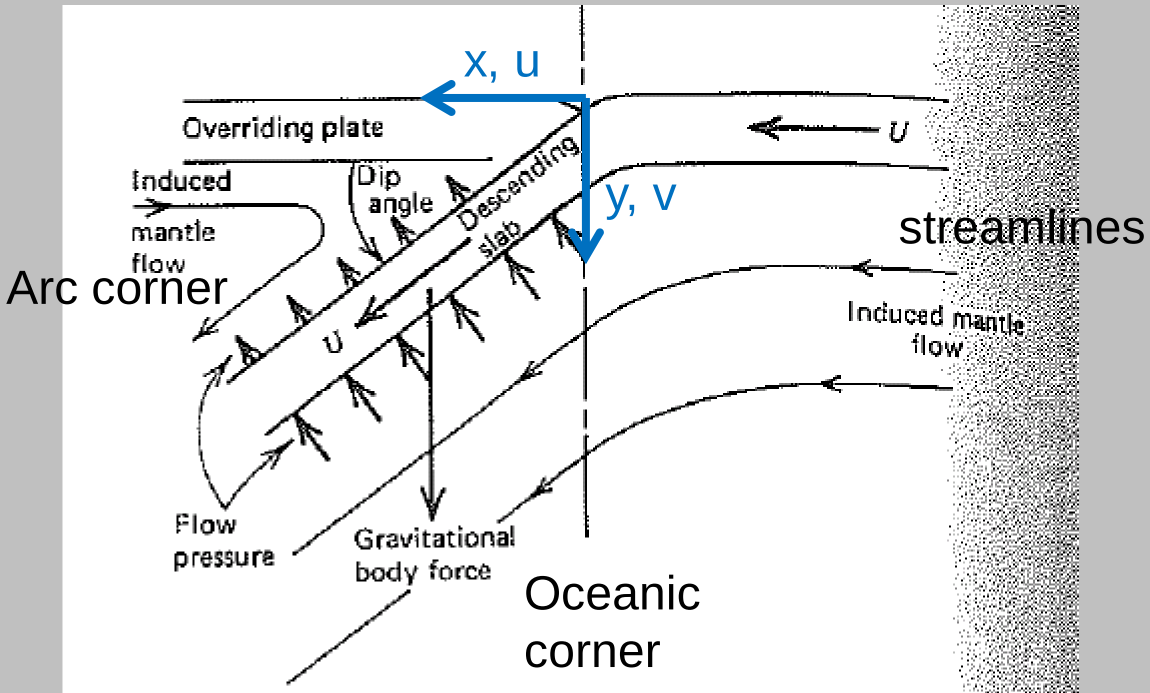
Where τ_r=relaxation time depends on viscosity and other parameters

Image of postglacial rebound

Mike Beauregard from Nunavut, Canada
http://en.wikipedia.org/wiki/File:Rebounding_beach,_among_other_things_(9404384095).jpg

Angle of Subduction

- Good example of using boundary conditions for a slightly more complex problem—now need to include gravity.



6–17 Forces acting on a descending lithosphere.

Balance of Torques from

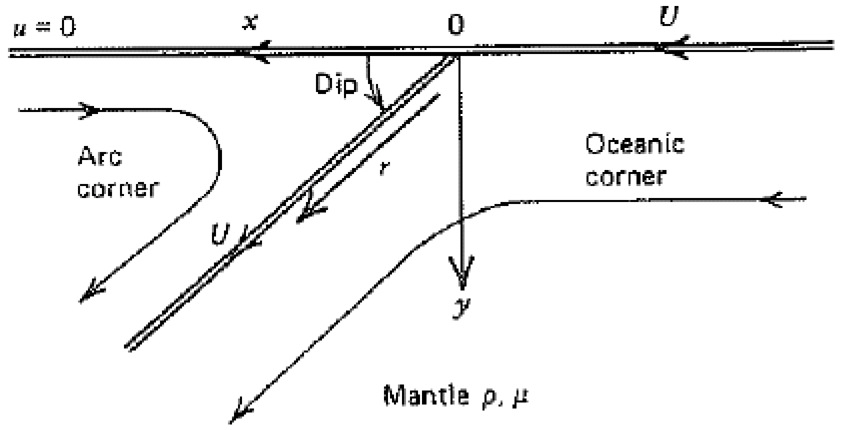
a) Gravity
b) Flow pressure induced by motion of descending lithosphere (trench suction)

Note tighter streamlines in corner due to geometry➔ pressure difference from bottom to top of slab. Also note that both top & bottom flow pressures are in same direction.

Also—after calculations, top exerts more torque than bottom (similar to why airplanes fly)

Angle of subduction

6–18 Viscous corner flow model for calculating induced flow pressures on a descending lithosphere.



Coordinate system

- Solution to continuity equation is the Biharmonic equation

$\nabla^4 \psi = 0$

Assume sol'n:

$$\psi = (Ax + By) + (Cx + Dy)\arctan\left(\frac{y}{x}\right)$$

Plug into eqn and show that it works

- Use eqns that we learned this week to take derivatives of ψ to get u and v, and pressures τ from the flow.

$$u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x}$$

EQ 6-1

$$\tau = \mu \frac{du}{dy}$$

Don’t forget!

- Derivatives of tan and arctan
- Torque = Force x Distance (cross product—or take moment arm from perpendicular)
- Too hard to do general case—book does specific case of dip=45 degrees—you will do dip = 60 degrees.

Diapirs (Rayleigh-Taylor Instabilities) (not nappies)

- Driven by gravity and density imbalances—high over low
- Examples:
  - Paint dripping
  - Mantle “drips”
  - Start of convection, plumes, lava lamps
  - Salt domes
- Could grow exponentially until it breaks up, or could die out--returning to original state (but not periodic—not elastic)

Basic Eqn: Incompressible continuity Eqn

$$\vec{\nabla} \cdot \vec{u} = 0 \text{ or } \nabla^4 \psi = 0$$

Balance Buoyancy Forces by Pressure Forces:

$$\vec{\nabla} P = \vec{\nabla}(p - \rho g y)$$

P=Pressure generated by flulid flow

p=pressure

Buoyancy=ρgy

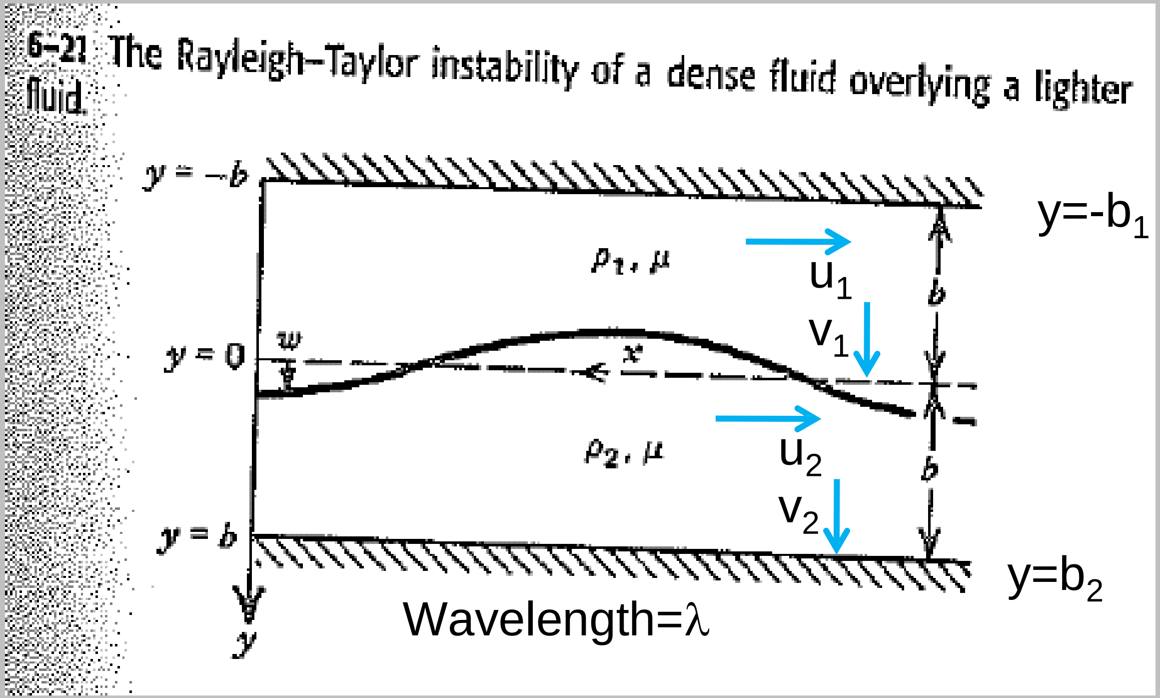
$$\vec{\nabla} P = \mu \nabla^2 u$$

(6-67 to 6-68)

=0 if forces are in balance (e.g., eqn 6-151)

To solve eqn—introduce stream function ψ

Like postglacial rebound or subducting plate—but boundary conditions differ



In general, $b_1 \neq b_2$

Displacement $w \ll b_1$ and b_2 -- approximation is very important –i.e., Interface shape is $w = A\cos 2\pi x/\lambda$

Because A is small, can treat interface as if it were at y=0 for the purposes of solving boundary conditions

- Boundary conditions:
  - 1)Rigid at top and bottom ($-b_1$ and b_2)—no slip condition (u continuous)

  $\therefore u=v=0$ at $y= -b_1$ and b_2
  - 2) Displacements and velocities and shear stress must be continuous across boundary between media (i.e., at interface, but since w is small, effectively y=0 here)

Guess solutions of ψ

- ψ_1; ψ_2 separate for each of top, bottom.
- ψ is similar in form to postglacial rebound, but uses hyperbolic functions instead of simple sines and cosines:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

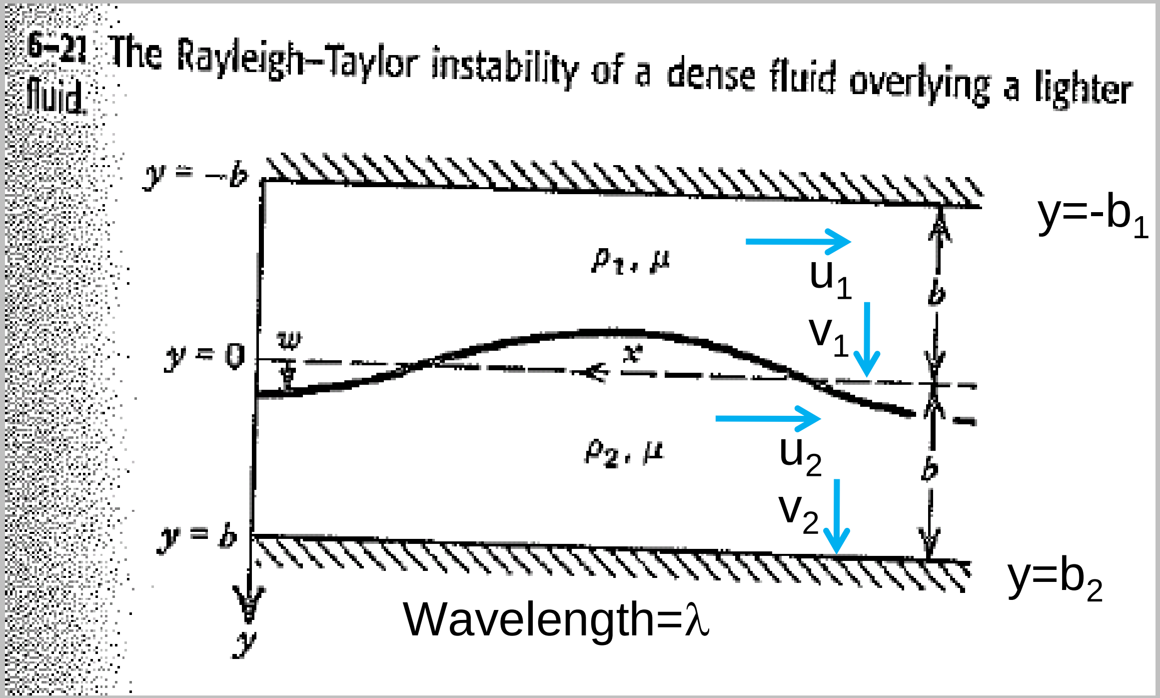
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\psi_1 = \sin\frac{2\pi x}{\lambda}\left(A_1 \cosh\frac{2\pi y}{\lambda} + B_1 \sinh\frac{2\pi y}{\lambda} + C_1 y \cosh\frac{2\pi y}{\lambda} + D_1 y \sinh\frac{2\pi y}{\lambda}\right) \quad (6-125)$$

(similar expression for ψ_2)

Solve by:

- Show that both $\psi_{1,2}$ are solns by substituting back into eqn,
- Determine $u_{1,2}$ and $v_{1,2}$ from derivatives of $\psi_{1,2}$
- $$u_{1,2} = -\frac{\partial \psi_{1,2}}{\partial y}; v_{1,2} = \frac{\partial \psi_{1,2}}{\partial x}$$
- Boundary conditions:
- u=v=0 at y= $-b_1$ and b_2 ➔ u(x,y) become $u_1(x,-b_1)=0$; $v_1(x,-b_1)=0$ $u_2(x,b_2)=0$; $v_2(x,b_2)=0$



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2) velocities and shear stress must be continuous across boundary between media (i.e., at y=0 here because w is small)

- $u_1(x,0)=u_2(x,0)$; $v_1(x,0)=v_2(x,0)$

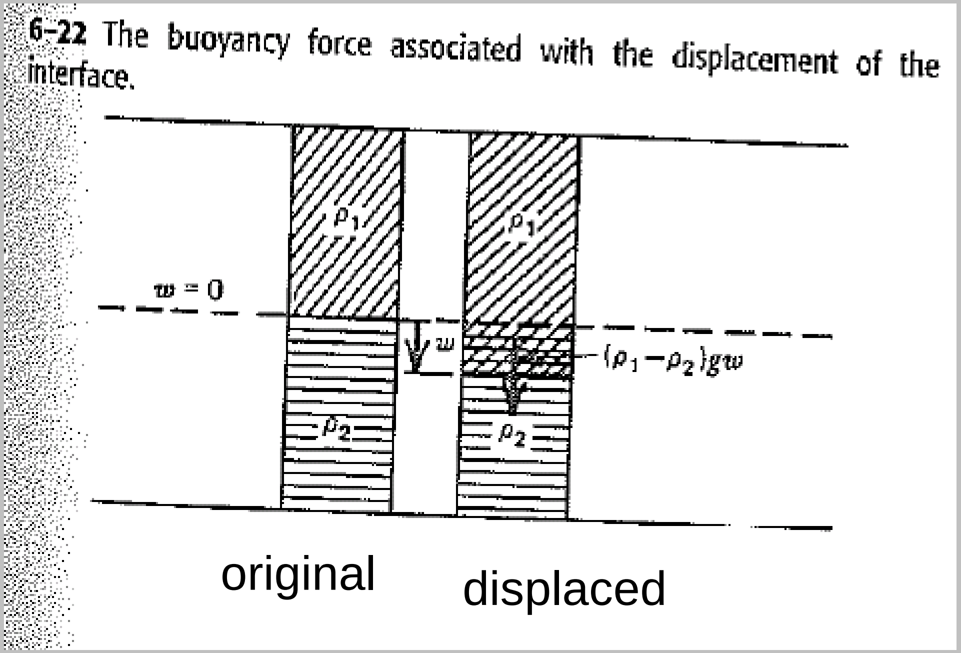
$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \text{ is same at boundary,}$$

$$\mu\left(\frac{\partial u_1(x,0)}{\partial y} + \frac{\partial v_1(x,0)}{\partial x}\right) = \mu\left(\frac{\partial u_2(x,0)}{\partial y} + \frac{\partial v_2(x,0)}{\partial x}\right)$$

- (x dependence is purely a function of $\sin(2\pi x/\lambda)$)
- Another key—interface is moving with the same velocity as the fluid, so at y=0

$$\frac{\partial w}{\partial t} = v(x,0)$$

Finally, balance forces--buoyancy and fluid flow pressure



$$(\rho_1 - \rho_2)gw = (P_2 - P_1) \text{ at } y = 0$$

Flow pressure found from integrating 6-72

Buoyancy

$$\frac{\partial P}{\partial x} = -\mu\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right)$$

Final solution after much algebra:

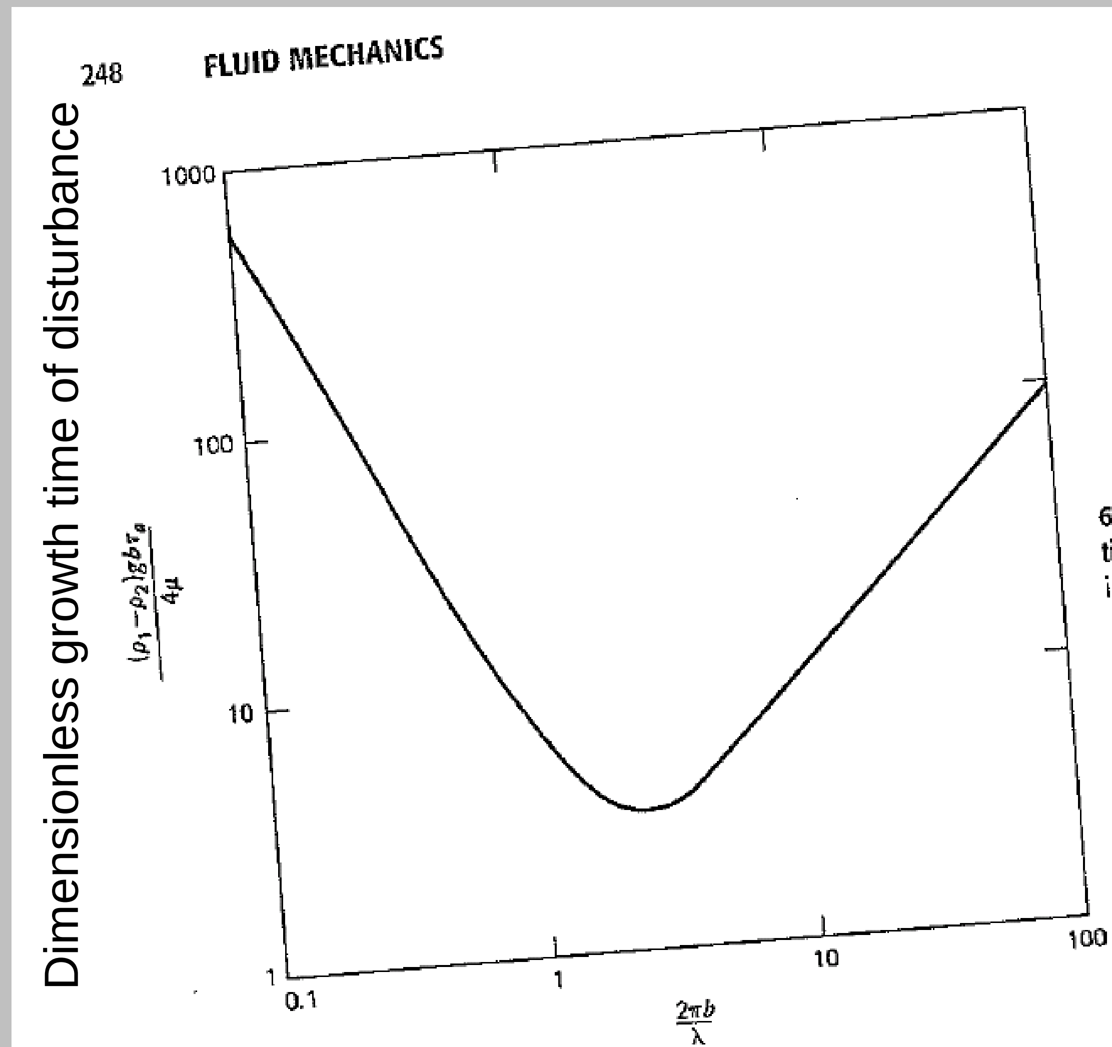
- Solution:

$$w = w_0 e^{t/\tau_a}$$

- Where τ_a is the growth time of the disturbance
- Is a function of sinh, cosh($2\pi b/\lambda$) multiplied by

$$\frac{4\mu}{(\rho_2 - \rho_1)gb}$$

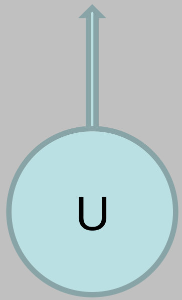
- τ_a depends on wavelength, but if have displacements at multiple wavelengths, fastest growing wavelength will dominate (τ_a is a minimum)



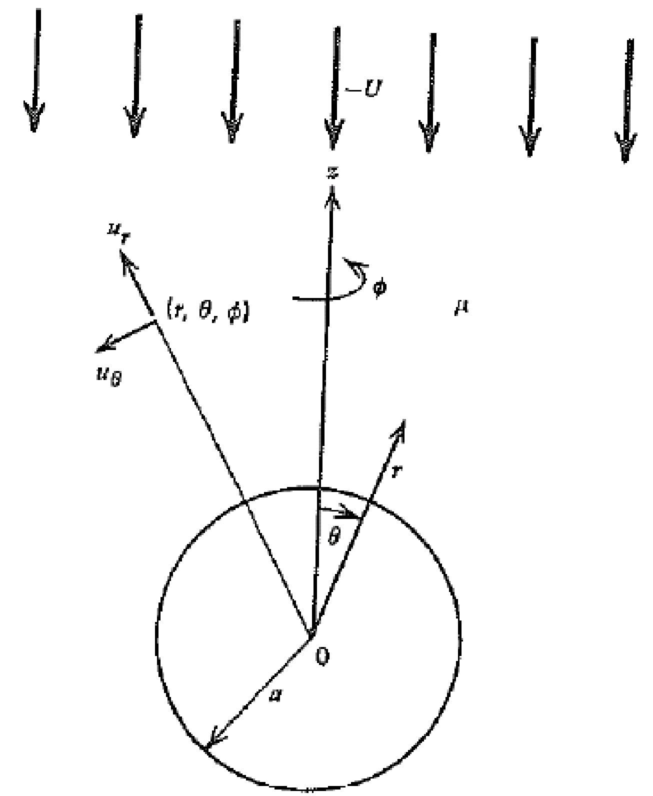
Dimensionless wavenumber

Stokes' Flow: How fast does a body fall due to its own weight?

- Applies in limit of very viscous fluid, with Re<1 (reversible flow)
- Applications:
  - Fall of pieces of slab
  - Rise of plumes/magma
  - Fall of metal probe

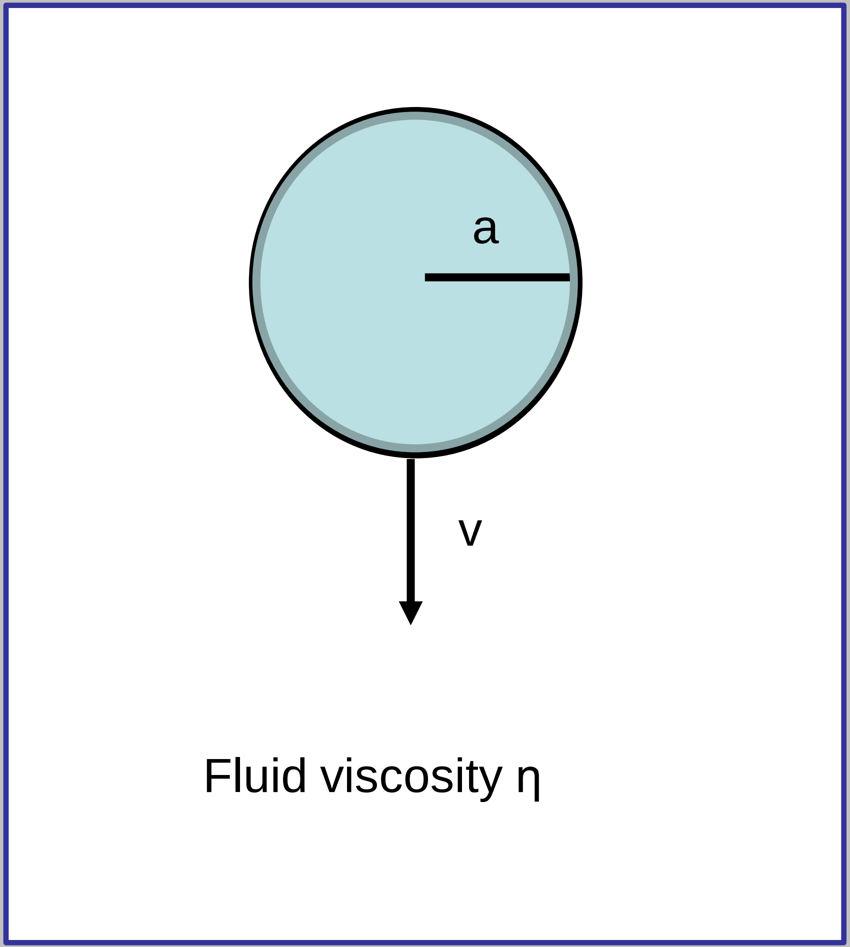


Ball rises through stationary fluid or fluid flows past stationary ball

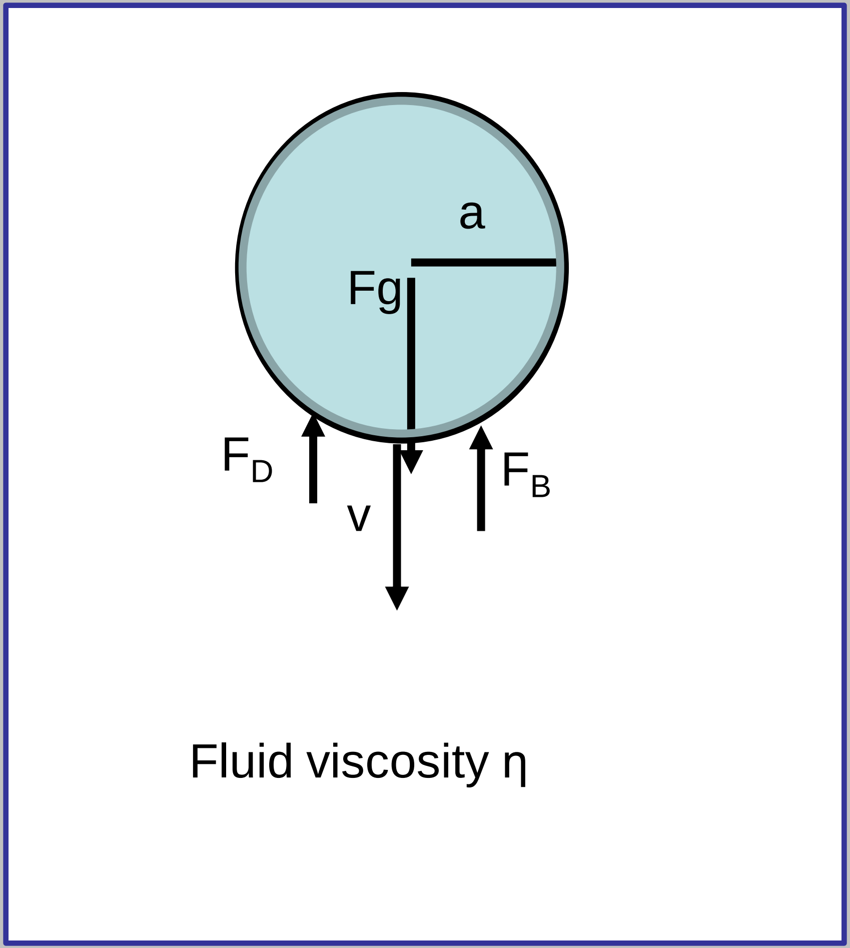


6–31 Steady flow of a viscous fluid past a sphere.

Sphere Falling in a Fluid

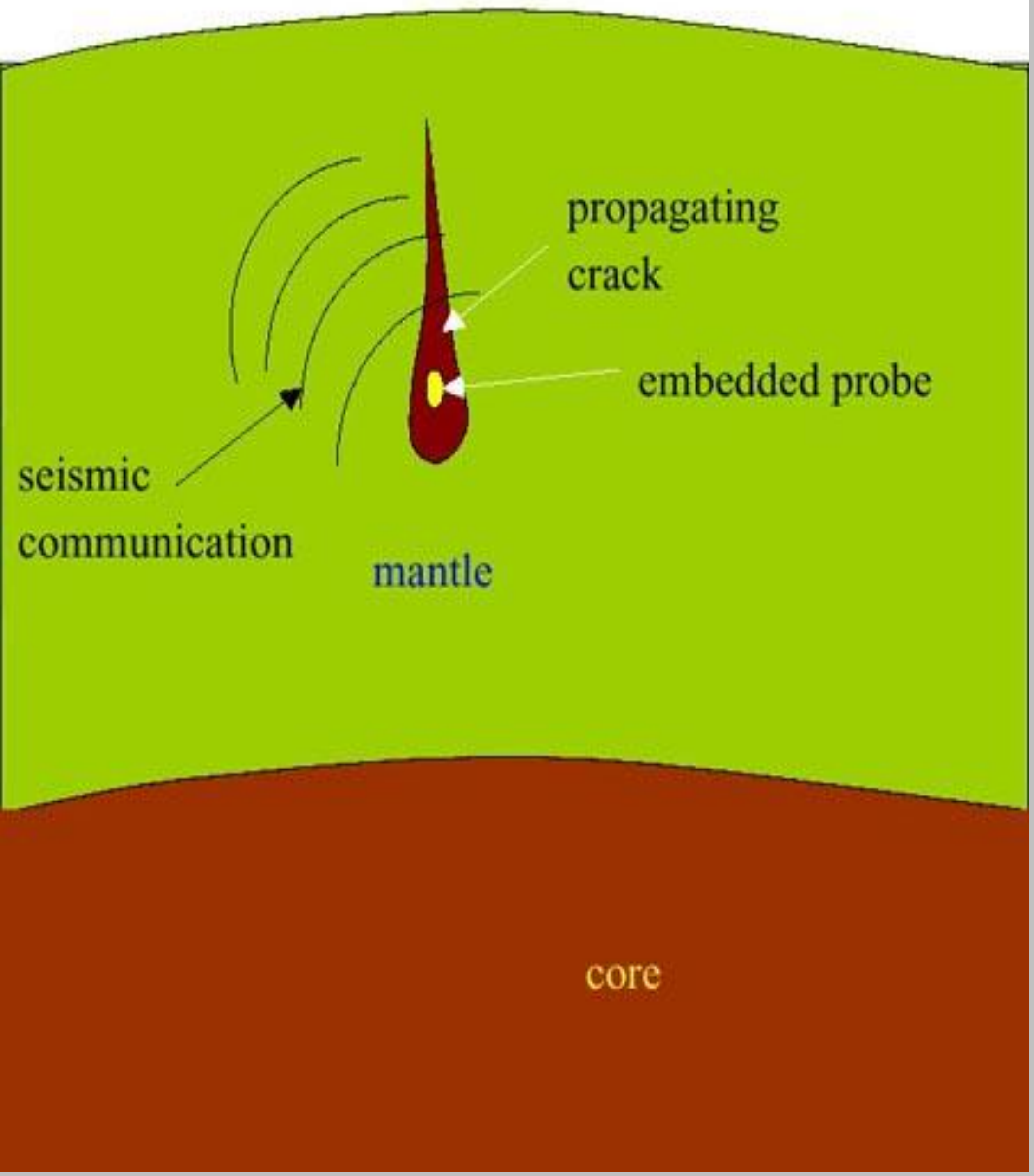


Sphere Falling in a Fluid



$F_g + F_B + F_D = 0$

Fall of Iron into Core



Stevenson, David J. Mission to Earth’s Core -A Modest Proposal. Nature, 423, 239-240, 2003.

About 1 week to get to core

Balance gravity (Buoyancy) and Viscous drag forces

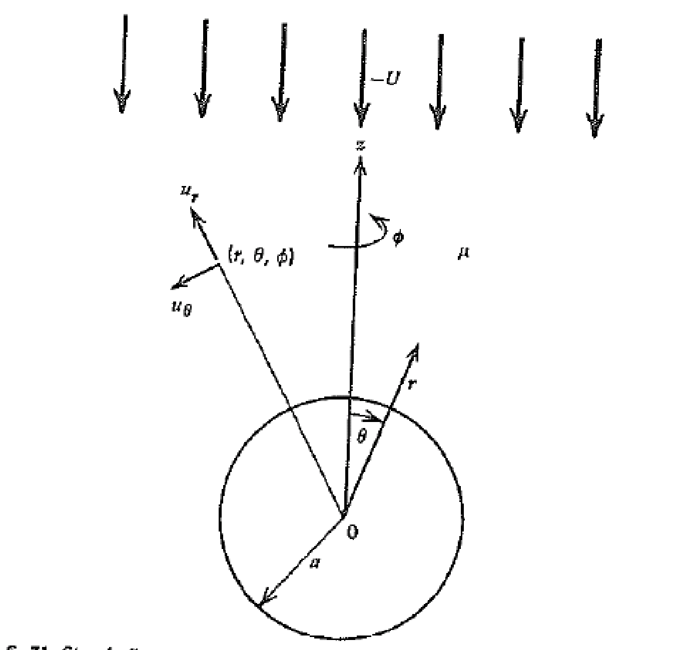
- Dominant equations: continuity equation and pressure equation again, same as before but now geometry and boundary conditions change

$$\vec{\nabla}\cdot\vec{u}=0 \qquad \vec{\nabla}P=\mu\nabla^2 u$$

- Where P=p-ρgy

- ρ_f=density of fluid
- ρ_s=denisty of sphere

$$\mathrm{Re}=\frac{\rho_f U(2a)}{\mu}$$



6–31 Steady flow of a viscous fluid past a sphere.

Boundary Conditions

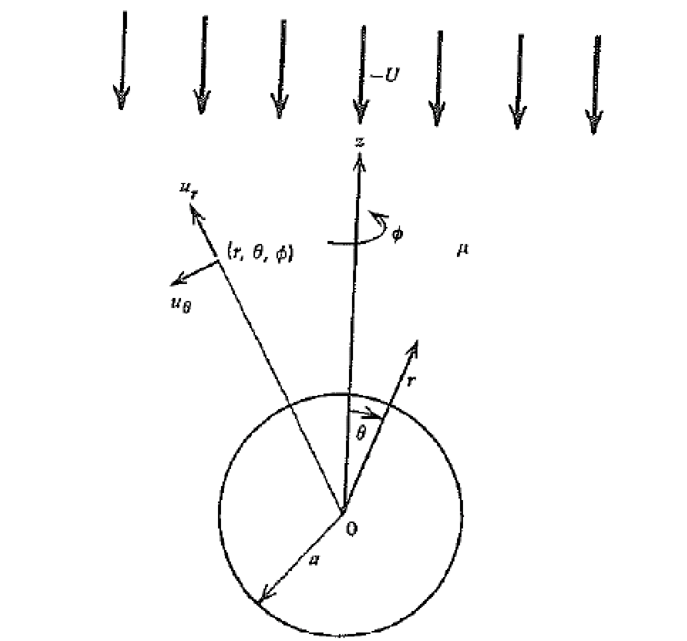
- As $r \rightarrow \infty$

  $u_r \rightarrow -U$ in z direction

  $u_r \rightarrow -U\cos\theta$ $u_\theta \rightarrow U\sin\theta$

No-slip on sphere: at r=a

  $u_r = u_\theta = 0$



6–31 Steady flow of a viscous fluid past a sphere.

Spherical Coordinates:

Continuity equation becomes:

$$0 = \vec{\nabla} \cdot \vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + (\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi})$$

But since u_Φ=0, last term is 0

To solve equation, also need the Laplacian of u:

$$\nabla^2 \vec{u} = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{u})$$

$$\vec{\nabla}\times\vec{u} = \frac{1}{r\sin\theta}[\frac{\partial}{\partial\theta}(u_\phi \sin\theta) - \frac{\partial u_\theta}{\partial\phi}]\hat{r} + \frac{1}{r}[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial(ru_\theta)}{\partial r}]\hat{\theta} + \frac{1}{r}[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}]\hat{\phi}$$

Pressure forces: Terms in P

Viscous forces: Terms in $\mu\nabla^2 u$

Solution

- Surprisingly, most terms drop out and …
- Pressure due to fluid flow is (Eq 6-216):

$$p = \frac{3\mu a U}{2r^2}\cos\theta$$

- Integrate to get downward "drag" (force) due to fluid pressure across sphere: $D_p = 2\pi\mu aU$

Viscous drag:

- Using 3-D formulation of stress again:

$$\overleftrightarrow{\tau} = \mu(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^{T})$$

Integrate to get Viscous Drag D_v=4$\pi\mu$aU

So total Drag F_D= Viscous Drag + Pressure Drag = D_p+ D_v=6$\pi\mu$aU

Speed of rise or fall:

- Balance Buoyancy Forces with Drag forces for steady-state case (no acceleration):
- $F_B=(\rho_f - \rho_s)g4\pi a^3/3 = F_D = 6\pi\mu aU$
- Solve for U
- For faster flow, Re>1, more difficult: use dimensionless drag coefficient C_D

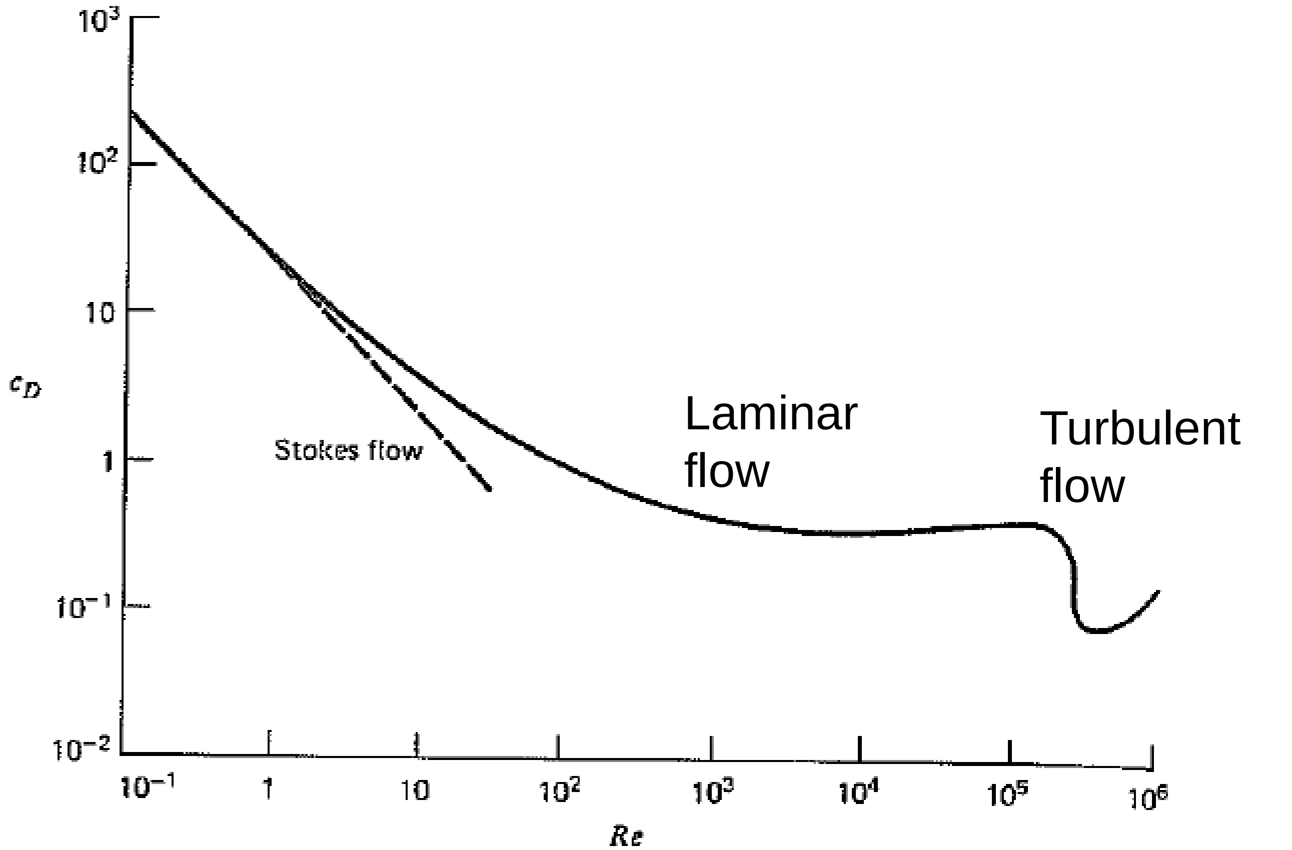
$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho_f U^2 \pi a^2} = \frac{24}{\mathrm{Re}} (6-226)$$

Pressure due to vel.

Sphere x-sec area (shadow)

- Stokes flow:

$$U = \frac{2(\rho_f - \rho_s) g a^2}{9\mu} (6-229)$$



- Re>1:

$$U = \left[\frac{8(\rho_f - \rho_s) g a}{3 C_D \rho_f} \right]^{1/2} (6-230)$$

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho_f U^2 \pi a^2} = \frac{24}{\text{Re}} (6-226)$$

Note—units work out in both cases