

School of Geography, Environmental and Earth Sciences and School of Mathematics,
Statistics and Computer Science

MATH 322/323, GPHS 323
Module on Fluid Flow in Earth Systems
2014 Computer Section Notes-Part 3 and Assignment 6

Due 5 PM Fri. 17 October 2014

Section 3 of Maths/GPHS 323 Fluid Flow project: Numerical models for more than one variable.

In your last project you solved several 1D numerical models in the steady state. This time, we will add the time dimension to a 1D numerical model and investigate 2D numerical models for heat flow and fluid flow.

Background: Boundary Conditions

For most of the differential equations we've studied, there are two types of boundary conditions.

- A) A defined value at a boundary. An example of this type of boundary is when we set the temperature at the top of the lithosphere equal to zero, or we set the velocity at the top of the asthenosphere equal to a fixed plate velocity.
- B) A derivative is constant at a boundary. An example of this type of boundary is the insulating boundary conditions for temperature problems, or the zero stress boundary conditions for fluid flow, where the derivative is equal to zero at a boundary.

You cannot solve a numerical problem without giving a value for the boundary conditions (or else, the computer will set the boundary to whatever it finds in its register, which can give very weird results that might vary each time you run it).

Setting the boundary conditions in the type of spreadsheets we are writing is actually quite straightforward.

C) Of course, defining a constant value is done just by not allowing it to vary, for example, for a 1-D case, you would set the top and the bottom of your column of temperature or velocity values equal to a constant value.

D) Setting a derivative equal to zero at a boundary is only slightly more complicated. For a 1-D case, simply set the top two values or bottom two values in your column equal to each other (e.g., set $E1=E2$). But unless you also have a condition on the top or bottom boundary, don't set $E1$ equal to anything in particular. Your previous spreadsheets have already implemented method 1) above, and the spreadsheet you are given here implements method 2).

Assignment:

For each section, include plots and print out your examples, and provide a discussion of what you have learned. You may hand in spreadsheets via email as supporting material, but you will be marked mainly on your printed version (legible handwritten sections are fine).

1. Finite difference calculations are often used to calculate the change in quantities in time, and 1-D heat flow is a popular application. Your notes contain an article from Thorston Becker that discusses how to solve the heat flow equation. This equation is:
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + \frac{H}{\rho c_p}$$

Where T is temperature, t is time, κ is thermal diffusivity, H is volumetric heat production, ρ is density, and c_p is heat capacity.

- a) **Show that** if we assume no heat production ($H=0$), we can rewrite the equation as

$$T(x_i, t_{j+1}) = T(x_i, t_j) + \kappa \Delta t \frac{(T(x_{i+1}, t_j) - 2T(x_i, t_j) + T(x_{i-1}, t_j)))}{\Delta x^2} \quad (\text{Becker's equation 11})$$

Note that we have assumed that T depends on a single dimension of space and on time, but that κ is constant. An example might be an insulated rod that is heated at one end and cooled at the other. [5 marks]

- b) Create a spreadsheet to calculate the temperature distribution in such a rod as a function of time. Use an initial model with a constant temperature except that one end of the model has three cells with a higher temperature than those around them (e.g., you might use $T=300\text{K}$ everywhere except in the 3 end cells, $T=600\text{K}$). Include the boundary condition that each end of the model has a constant temperature. Hint: The main difference between this problem and the one for 1-D fluid flow is that you will be solving a non-steady-state problem, so instead of each iteration being a better estimate of the final solution, this time each iteration should give you a new temperature profile for a later time. Set up one column as an initial temperature model similar to the initial viscosity model in previous weeks' spreadsheets. A second column will give the updated temperatures. You will need to set up the temperature column so that the new temperature is equal to its previous value plus some other terms depending on the value of the terms around it. (e.g., cell d10 might have the value $=d10+(\text{other terms in d11, d12, and d10 again})$). If you get to the point where you can see the temperature anomaly spread out over time with increasing numbers of iterations, you will have done well.

(Hint 2: To get the spreadsheet to work, you will need to satisfy the stability criterion:

$$\Delta t \leq \frac{(\Delta x)^2}{2\kappa} . \text{ Physically, this means that the time step is smaller than the time it takes for}$$

heat to diffuse across a single block.) Find the thermal constant for metal and determine the actual time expected to diffuse heat across a one meter rod. Show the time it takes for the heat to get to 400 K in the middle of the rod. Show graphs of your solutions, and explain them. [10 marks]

- c) Try the spreadsheet for a typical value of $\kappa = 1\text{e-}6$ for the mantle, and typical dimensions, for, e.g., a plug of magma that suddenly rises and heats a column of rock above it. Suppose the magma is at 20 km depth. Plot graphs of the temperature distributions at three times—once at the start, once when the temperature change has affected about half the length of the material, and once when the temperature is approaching steady state. Determine what those times are in years. [5 marks]

d) For your spreadsheet, try a value of Δt that is too large for the stability criterion. (Try it as two times too big and 10 times too big). Discuss what happens when you iterate, showing at least one figure for proof. [5 marks]

2. Now we turn to solving the diffusion equation in two dimensions. We will again start out with a very simple Excel spreadsheet. It is the attached file: heatflow_truefd_toclass.xls

It calculates a solution to the diffusion equation in two dimensions, neglecting transport, i.e.

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

It is also solved iteratively with the finite difference scheme. This time there are two spatial dimensions, and a time derivative as well.

If we set our grid such that our x and y grids have the same dimension, the finite difference calculation becomes, for each element T_{jk} and the nth time step,

$$T_{jk}^{n+1} = T_{jk}^n + \kappa \Delta t \frac{(T_{j+1,k}^n + T_{j-1,k}^n + T_{j,k+1}^n + T_{j,k-1}^n - 4T_{j,k}^n)}{\Delta x^2}$$

The boundary conditions for this problem have been set so that the sides of the model are given by the initial model (Columns N and Y; constant in this case). The top and bottom rows are equal to the third & 3rd to last rows, respectively, which is a way of setting the gradient equal to zero in those places, i.e., like the insulating boundary conditions we've seen in the analytic solution for the centre of the model, away from the boundaries.

The spreadsheet plots the initial values, the final values, and a profile through the centre of the model. It is another "brute force" solution, but is stable providing certain conditions are met. An example of the kind of thing this model might represent is a metal plate onto which you put hot food.

By using the spreadsheet and plotting at several sets of times, show that the model has heat diffusing outward consistently with time and distance, and that the anomaly decreases over time. Play with the initial conditions and the parameters mesh and kappa and show what happens when the stability condition is not satisfied (i.e., when the parameter $\gamma < 2$). More sophisticated schemes exist that use a damping parameter to help to get the program to converge, but we will not discuss these this year. [5 marks]

3. Your notes contain a paper by David Eaton and Andrew Frederiksen published a few years ago in Nature. They perform a finite-difference calculation of two-dimensional heat flow in response to a plume source. Modify the values in the spreadsheet and show that you get a similar order of magnitude response when you put in a similar structure. (Hint: Eaton and Frederiksen include a temperature gradient in their model. You should use the same gradient in your model as well.) [10 marks]

4. The steady-state solution to the heat equation is found by setting the time derivative of temperature to 0. Thus, it becomes Laplace's equation, or $\nabla^2 T = 0$. This equation can also result from fluid flow in special circumstances. The one we will discuss here is that of the horizontal velocity $u(y,z)$ in the mantle beneath a transform (strike-slip) plate boundary, where the motion of the fault is in the x direction. Beneath a very long transform, the velocity components v and w vanish because u is independent of x (Figure 1). The equations of motion for three dimensional

viscous flow therefore reduce to $\frac{dp}{dz} = \rho g, \nabla^2 u = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, which is simply Laplace's equation

in two dimensions. If we solve for $u(y,z)$, assuming that the viscous mantle extends to great depth ($z \rightarrow \infty$) beneath the plate boundary, we find that $u_x = U(1-(2\theta/\pi))$, $u_y = 0$; $u_z = 0$., where θ is in cylindrical coordinates, as shown below (x is the long axis of the cylinder). The solution only varies in the $y-z$ plane. We assume the plate is rigid, and the bottom of the plate is at the origin of the coordinate system. The solution to this problem was published in a 2004 paper I wrote. This provides the analytic solution for testing against the numerical solution in the next section. The citation is: Savage M.K., K.M. Fischer, and C.E. Hall (2004), Strain modelling, seismic anisotropy and coupling at strike-slip boundaries: application in New Zealand and the San Andreas fault, in Vertical Coupling and Decoupling in the Lithosphere, 227, 9-39, Geological Society, London, Special Publications, on p. 9. I've attached the paper here.

Note that the solution to Laplace's equation is a simpler version of the heat flow equation, with no time dependence; i.e., it is the steady state solution. Thus, show that the new equation can be rewritten as:

$$u_{jk} = \frac{(u_{j+1,k} + u_{j-1,k} + u_{j,k+1} + u_{j,k-1})}{4}.$$

Thus, the value of a point is the average of all the points around it. Make a new spreadsheet to calculate the velocity u as a function of depth, given the boundary conditions shown in the model. (+U and -U on either side of the fault, $u=0$ at the bottom of the model and stress-free boundary conditions on the sides). Show that the results you get match the analytical solution. [10 marks]

5. Provide a discussion of what you have learned from the computer modeling section of the course. [10 marks]

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METHOD

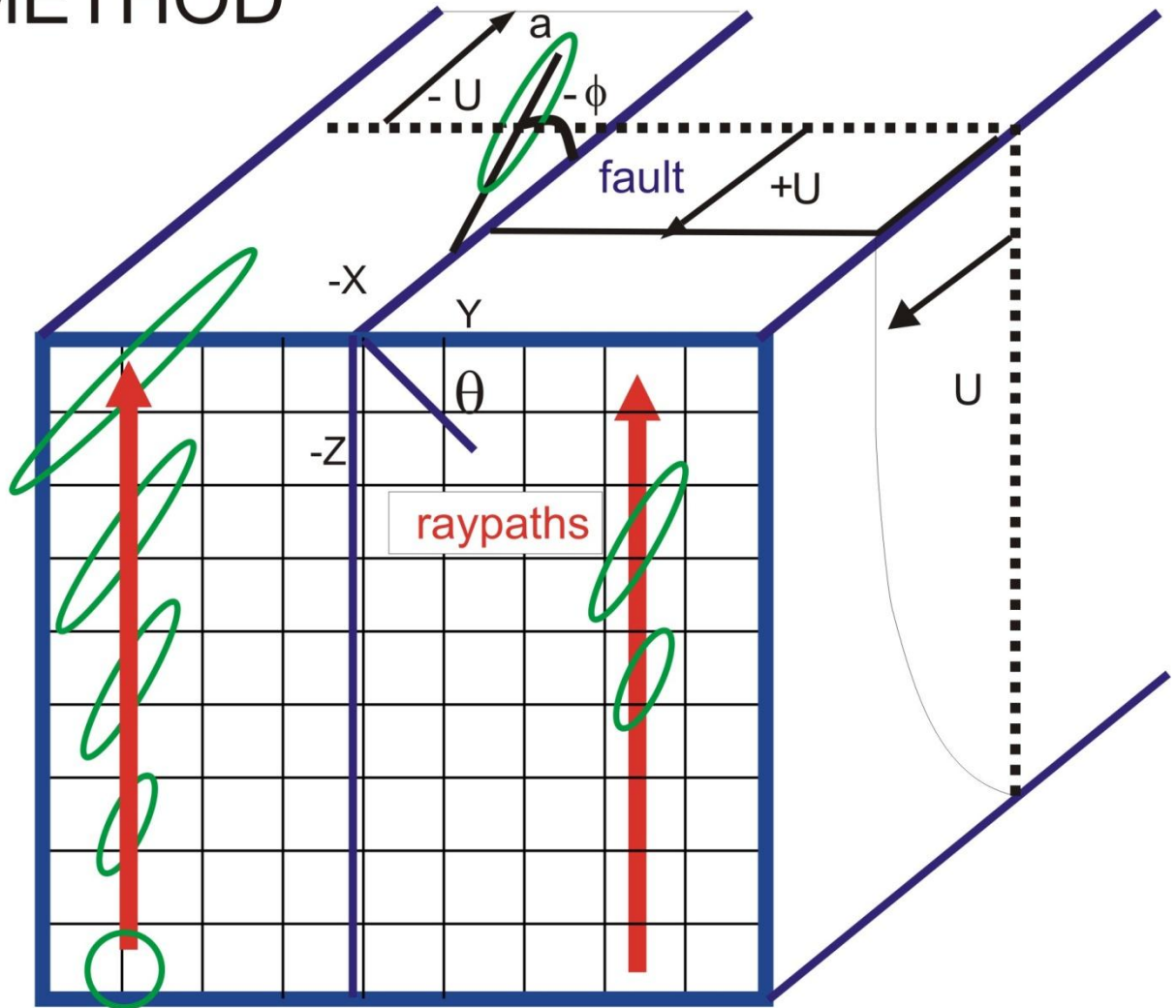


Fig. 1

Figure 1. A diagram of the model setup for determining mantle flow due to a strike-slip fault at the surface. The strike-slip fault is modeled as a break with the boundary condition that the velocities on either side of the fault are parallel to the fault and in opposite directions to each other.

References:

See some of the notes from previous times for explanations of diffusion equations. More details can be found in Chapters 5 and 6 of Mark Spiegelman's web page:

<http://www.ldeo.columbia.edu/~mspieg/mmm>