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MATH 321/322/323 Applied Mathematics T1 and T2 2013

Module on Mechanix: Assignment 2

- This second assignment will deal with the Lagrangian description of motion in a central potential.
- Read the chapters on the notes that deal with the Lagrangian formulation.
- Applied math level proofs are good enough — I do not demand absolute analytical rigour, but do want to see enough information so that any interested reader could fill in any missing steps.
- If you find any typos in the notes or assignment, please let me know.

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1. Consider a single particle with mass m and position $\vec{x}(t)$ that is acted on by a central potential

$$V(\vec{x}) = V(\,||\vec{x}||\,) = V(r).$$

That is, the potential $V(r)$ depends only on *distance from the origin.*

(a) First write down the Lagrangian $L(\dot{\vec{x}}, \vec{x})$ as a function of (Cartesian coordinate) position and velocity.

(Given what you have seen so far, this step is [or should be] trivial.)

(b) Now adopt spherical polar coordinates (r, θ, ϕ).

Warning: I will be using *physicists*' conventions for θ and ϕ, which are quite standard but are opposite to Anton's.

I have no idea where Anton's conventions came from.

If you insist on using Anton's conventions, be prepared for a few $\theta \leftrightarrow \phi$ interchanges below.

i. Calculate the position $\vec{x}$, or more precisely the Cartesian coordinates (x, y, z), as explicit functions of the spherical polar coordinates $(r,\ \theta,\ \phi)$.
(This is completely and utterly standard, at worst you will need to look it up somewhere.)

ii. Calculate the velocity $\dot{\vec{x}}$, or more precisely the Cartesian components $(\dot{x}, \dot{y}, \dot{z})$, as functions of $(\dot{r},\ \dot{\theta},\ \dot{\phi})$ and $(r,\ \theta,\ \phi)$.
(This might require you to do some chain rule differentiations, but it is not at all a difficult calculation.)

(c) Hence write down the kinetic energy

$$K(\dot{r}, \dot{\theta}, \dot{\phi};\ r, \theta, \phi)$$

as a function of the spherical polar coordinates $(r,\ \theta,\ \phi)$, and their time derivatives $(\dot{r},\ \dot{\theta},\ \dot{\phi})$.

(d) Now write down the Lagrangian

$$L(\dot{r}, \dot{\theta}, \dot{\phi};\ r, \theta, \phi)$$

as a function of the spherical polar coordinates $(r,\ \theta,\ \phi)$, and their time derivatives $(\dot{r},\ \dot{\theta},\ \dot{\phi})$.

(e) Separate the Lagrangian into two terms of the form

$$L(\dot{r}, \dot{\theta}, \dot{\phi};\ r, \theta, \phi) = L_{\text{radial}}(\dot{r};\ r) + r^2 X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi)$$

and explicitly evaluate both the radial $L_{\text{radial}}(\dot{r};\ r)$ and angular $X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi)$ terms.

Warning: Note that the quantity $X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi)$ is *not*, by itself, a Lagrangian, and trying to write down Euler–Lagrange equations for $X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi)$ will lead to abject nonsense...

(f) Write down the complete set of Euler–Lagrange equations for the (complete) Lagrangian L.

(There will be three Euler–Lagrange equations, one for r, one for θ, and one for ϕ.)

(g) Let us simplify life by assuming the motion lies purely in the equatorial plane of the spherical polar coordinates — that is take $\theta = \pi/2$ while the azimuthal angle $\phi(t)$ is allowed to vary as a function of t.

(**Comment:** This is not actually an *assumption*, it is merely a convenient *choice* of coordinates that implies no loss of generality — but it would take us too far afield to justify this fully.)

i. With this simplification — assuming equatorial motion, and starting from the 3 Euler–Lagrange equations derived above, write down the (simplified) equation of motion for ϕ.

ii. Also, assuming equatorial motion, and starting from these same 3 Euler–Lagrange equations derived above, write down the simplified equation of motion for r.

iii. (Trivial.) Just for completeness, under this assumption what happens to the Euler–Lagrange equation for θ?

(h) From this simplified equation of motion for $\phi(t)$, and an elementary geometrical argument, deduce that the line joining the *origin* with the *particle* at $r(t)$, $\theta = \pi/2$, $\phi(t)$, *sweeps out equal areas in equal times.*

(i) Deduce that Kepler's *second* law of planetary motion would hold for *any* force law described by a central potential.

(Comment: When discussing Kepler's laws, we are currently approximating the sun as fixed, and the point particle at $\vec{x}(t)$ as one of the planets; if we wanted to get a little fancier we could look at the 2-body sun-planet system and separate out the centre of mass and relative motions as in assignment 1 — we would have the same result — Kepler's second law implies and is implied by any arbitrary central potential.)

(j) From the above, what can you deduce about the angular momentum of the system?

2. Now continue with the system we have been considering above.

   (a) Consider the Lagrangian

   $$L(\dot{r},\dot{\theta},\dot{\phi};\, r,\theta,\phi) = L_{\text{radial}}(\dot{r};\, r) + r^2 X_{\text{angular}}(\dot{\theta},\dot{\phi};\theta,\phi)$$

   and specifically put in the constraint of equatorial motion, $[\theta = \pi/2]$, *at the level of the Lagrangian*.
   Show that one now has

   $$L(\dot{r},0,\dot{\phi};\, r,\pi/2,\phi) = L_{\text{radial}}(\dot{r};\, r) + r^2 X_{\text{angular}}(0,\dot{\phi};\pi/2,\phi)$$

   Evaluate X_{angular}.
   You should find that both L itself and X_{angular} are independent of ϕ so that one can write

   $$L \to L(\dot{r},\dot{\phi},\ r) = L_{\text{radial}}(\dot{r};\, r) + r^2 X_{\text{angular}}(\dot{\phi}).$$

   This has now effectively reduced everything to two-dimensional motion in the radial and azimuthal directions; motion in the equatorial (r,ϕ) plane.

   (b) Calculate the two remaining Euler–Lagrange equations coming this "reduced" two-dimensional Lagrangian.

   You will find two differential equations, one for $r(t)$ and one for $\phi(t)$, that are coupled to each other, and one of which is coupled to the potential $V(r)$.
   (Comment:
   These two-dimensional Euler–Lagrange equations should be the same as you obtained by first calculating the three-dimensional Euler–Lagrange equations and then going to equatorial motion. In this particular case the constraint $\theta = \pi/2$ is a simple algebraic constraint, so you can *either* enforce the constraint at the level of the Lagrangian and then deduce the Euler–Lagrange equations, *or* you could first calculate the Euler–Lagrange equations and then enforce the constraint at the level of the equations of motion. In this particular case it makes no difference.)

   (c) Now consider the special set of solutions to these ODEs (ordinary differential equations) of the form $r = r_0 =$ constant.

These correspond to circular motion.

Putting $r = r_0 =$ constant into the two Euler–Lagrange equations derived above, one becomes trivial, and the other will give you an explicit formula relating $\dot{\phi}$ to the radius r_0 and the gradient of the potential at r_0.

That is, for circular orbits there will be some function such that

$$\dot{\phi} = f\left(r_0, V'(r_0)\right).$$

Specifically, show that for circular orbits

$$\dot{\phi} = \sqrt{\frac{V'(r_0)}{mr_0}}$$

(d) Kepler's third law of planetary motion states that: "(the orbital period of the planet)^2 is proportional to (the semi-latus rectum of the orbit)^3."

Approximating planetary orbits by circles this means

$$(\text{period})^2 \propto (\text{radius})^3$$

But for circular orbits the period is easy to calculate in terms of the azimuthal angular velocity $\dot{\phi}$:

$$(\text{period}) = \frac{2\pi}{\dot{\phi}}$$

So Kepler's third law (for circular orbits) becomes

$$\left(\frac{2\pi}{\dot{\phi}}\right)^2 \propto r_0^3$$

Now combine Kepler's third law of planetary motion with the general formula for $\dot{\phi}$ we derived above for circular orbits in any central potential to deduce a scaling law for $V'(r_0)$ in our solar system:

$$V'(r_0) \propto\ ???$$

Integrating this, find a scaling law for the potential $V(r_0)$ in our solar system:

$$V(r_0) \propto\ ???$$

(e) Thereby deduce Newton's inverse square law from Kepler's laws of planetary motion.

(To really finish the job you would then have to check that the inverse square law still gives the right results for non-cirrcular orbits — you would first have to check that the inverse square law leads to elliptical orbits, and then check that Kepler's 3rd law still holds for these elliptical orbits. To solve this particular astronomical problem Newton had to *invent* the calculus.)

3. Now continue with the system we have been considering above (for general $V(r)$, not just inverse square law).

(a) Take the simplified equation of motion for $\phi(t)$, derived by assuming equatorial motion, and integrate it.
Prove that

$$\dot{\phi} = \Omega(r)$$

where $\Omega(r)$ is a *simple* and *explicit* function of r (plus an integration constant).
Explicitly evaluate $\Omega(r)$.

(b) Now consider the Lagrangian

$$L(\dot{r}, \dot{\theta}, \dot{\phi};\ r, \theta, \phi) = L_{\text{radial}}(\dot{r};\ r) + r^2 X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi)$$

and specifically put in the constraints of equatorial motion, $[\theta = \pi/2]$, *and* the result just obtained above, $[\dot{\phi} = \Omega(r)]$.
Show that one now has

$$L(\dot{r}, 0, \Omega(r);\ r, \pi/2, \phi) = L_{\text{radial}}(\dot{r};\ r) + r^2 X_{\text{angular}}(0, \Omega(r); \pi/2, \phi)$$

Evaluate L_{angular}.
You should find that both L itself and L_{angular} are independent of ϕ so that one can write

$$L \to L(\dot{r},\ r) = L_{\text{radial}}(\dot{r};\ r) + r^2 X_{\text{angular}}(\Omega(r)).$$

This has now effectively reduced everything to simple one-dimensional motion in the radial direction — all the angular dependence is now hiding in the single function $\Omega(r)$.

(c) Re-group the Lagrangian into the form

$$L = \frac{1}{2} m \, (\dot{r})^2 + V_{\text{effective}}(r)$$

and explicitly evaluate $V_{\text{effective}}(r)$ in terms of $V(r)$, $\Omega(r)$, and r itself.

(d) Calculate the one remaining Euler–Lagrange equation for this "reduced" one-dimensional Lagrangian.

You will find a single differential equation for $r(t)$ in terms of some combination of the potential $V(r)$ and the angular motion $\Omega(r)$, (and r itself).

4. Now continue with the system we have been considering above (again for general $V(r)$, not just inverse square law).

(a) Under what conditions is

$$X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi) = 0?$$

(This can be viewed as a purely mathematical question.)

(b) Assuming that at some initial time $X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi) = 0$, use the Euler–Lagrange equations to show that this quantity will *remain zero* for all time.

(c) Physically interpret this situation.

(d) Assuming that $X_{\text{angular}}(\dot{\theta}, \dot{\phi}; \theta, \phi) \equiv 0$ for all time, what does the Lagrangian L reduce to?

Find the one remaining the Euler–Lagrange equation.

Interpret.

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