MATH/GPHS 322/ 323 Cartesian Tensors Module

Chapter 2

Contents:

Real symmetric Matrices

To understand the importance of symmetry of a tensor we must make use of one of the most useful theorems in linear algebra. We will deal with it in a general form.

Definition: We extend the concept of orthogonal matrix already developed for 3 x 3 matrices: If A is a real square matrix (NxN) with the property that

$$A\,A^T = A^T A = I_N$$

Then we say A is an orthogonal matrix. (The definition is also extendable to complex matrices.)

As before, A^T is the inverse of A.

Theorem: If a matrix E is a real, symmetric (NxN) matrix, there exists an orthogonal matrix A (NxN) such that $A^T E A = \Lambda$, where Λ is a diagonal matrix, viz:

$$\Lambda = \begin{bmatrix} \lambda_1, 0, 0, \ldots \\ 0, \lambda_2, 0, 0, \ldots \\ 0, 0, \lambda_3, 0, 0, \ldots \\ \\ \\ \ldots\ldots\ldots 0, 0, 0, \lambda_N \end{bmatrix}$$

Outline of Proof

The proof follows from a long chain of sub-theorems. We outline them.

We begin by looking at the Eigenvalues (λ) and Eigenvectors ($\underline{\alpha}$) of E. Recall that these are defined by the equation:

$$E\,\underline{\alpha} = \lambda\,\underline{\alpha} \qquad (1)$$

The condition that there should exist non-trivial eigenvalues and eigenvectors for E is found as follows. Write eqn 1:

$$E\,\underline{\alpha} - \lambda I \underline{\alpha} = (E - \lambda I)\,\underline{\alpha} = 0$$

Treat $(E - \lambda I)$ as a vector of columns, and multiply this out:

$$\sum_{i=1}^{N} \alpha_i .\{\text{column}_i \text{ of } (E - \lambda I)\} = 0$$

That is, the column rank of $(E - \lambda I)$ is less than N; i.e. $(E - \lambda I)$ is singular. Therefore its determinant must be zero.

$$|(E - \lambda I)| = 0 \qquad (2)$$

Now expanding this determinant gives a polynomial of order N in λ. The fundamental theorem of algebra says that this equation has N (possibly complex) roots, not all of which need be distinct.

We can write this polynomial: $(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_N) = 0$

[NB 'not distinct' means that some of the λ_i may be repeated.]

Now for each λ_i we can solve for $\underline{\alpha}_i$:

$$E\,\underline{\alpha}_i = \lambda_i\,\underline{\alpha}_i \qquad (1a)$$

Note that if $\underline{\alpha}_i$ satisfies eqn 1a, then so does $k\,\underline{\alpha}_i$. Therefore, to uniquely define the eigenvectors, we adopt the convention that they are of *unit length*.

(i) If E is real and symmetric, the λ_i are all real.

From eqn 1a,

$$\underline{\alpha}_i^{H} E \, \underline{\alpha}_i \quad = \quad \lambda_i \underline{\alpha}_i^{H} \underline{\alpha}_i = \quad \lambda_i \text{, because } \underline{\alpha}_i \text{ is unit length.}$$

(where the ^{H} denotes the *Hermitian* - i.e. complex conjugate transpose - of the eigenvector).

Take the Hermitian of the equation:

$$\underline{\alpha}_i^{H} E^{H} \underline{\alpha}_i \quad = \quad \underline{\lambda}_i \, .$$

where $\underline{\lambda}$ means complex conjugate.

But E is real and symmetric, so $E^H = E$, so

$$\underline{\alpha}_i^{H} E^{H} \underline{\alpha}_i \quad = \underline{\alpha}_i^{H} E \, \underline{\alpha}_i$$

which means $\lambda_i = \underline{\lambda}_i$; i.e. λ_i is real.

(ii) If E is real and symmetric, then there is a real eigenvector for every eigenvalue.

From $(E - \lambda I) \underline{\alpha} = 0$, since $(E - \lambda I)$ is real, it follows that the real part of $\underline{\alpha}$, $\underline{\alpha}_R$, must satisfy $(E - \lambda I) \underline{\alpha}_R = 0$. I.e. $\underline{\alpha}_R$ is a real eigenvector. Since everything is now real, we can now revert to ^{T} for transpose.

(iii) If E is real and symmetric, the eigenvectors are mutually orthogonal.

First, consider eigenvectors $\underline{\alpha}_i$ and $\underline{\alpha}_j$ associated with distinct eigenvalues λ_i and λ_j.
From eqn 1a,

$$\underline{\alpha}_j^{T} E \, \underline{\alpha}_i \quad = \quad \lambda_i \underline{\alpha}_j^{T} \underline{\alpha}_i$$

and

$$\underline{\alpha}_i^{T} E \, \underline{\alpha}_j \quad = \quad \lambda_j \underline{\alpha}_i^{T} \underline{\alpha}_j$$

Take the Hermitian of the second equation, remembering that the eigenvalues are real:

$$\underline{\alpha}_j^{T} E^{T} \underline{\alpha}_i \quad = \quad \lambda_j \underline{\alpha}_j^{T} \underline{\alpha}_i$$

But E is real and symmetric, so $E^T = E$. Taking differences:

$$\underline{\alpha}_j^{T} E^{T} \underline{\alpha}_I - \underline{\alpha}_j^{T} E \, \underline{\alpha}_i = 0 = \lambda_j \underline{\alpha}_j^{T} \underline{\alpha}_i - \lambda_i \underline{\alpha}_j^{T} \underline{\alpha}_i = (\lambda_j - \lambda_i) \underline{\alpha}_j^{T} \underline{\alpha}_i$$

But $(\lambda_j - \lambda_i) \neq 0$ because they are distinct eigenvalues. Therefore $\underline{\alpha}_i^{T} \underline{\alpha}_j = 0$, i.e. the eigenvectors are orthogonal.

Second, consider the case where some of the eigenvalues are repeated. For each repeated eigenvalue it can be shown that there is a subspace of R^N of dimension equal to the number of times the eigenvalue is repeated, in which every vector is an eigenvector corresponding to that repeated eigenvalue. So e.g. if an eigenvalue is repeated 3 times, there is a space R^3 in which every vector is an eigenvector for that eigenvalue. The subspace is orthogonal to the subspaces corresponding to the other eigenvalues - because we have proved that distinct eigenvalues have orthogonal eigenvectors. In the subspaces, we can find as many orthogonal vectors as the dimension of the subspace. Therefore we can find a set of N mutually orthogonal, real eigenvectors for every set of N eigenvalues, but they will not be unique if there are repeated roots to $|(E - \lambda I)| = 0$.

Now write $A = (\underline{\alpha}_1, \underline{\alpha}_2, \ldots \underline{\alpha}_N)$

i.e. construct a matrix by using the eigenvectors of E as columns. A is orthogonal by the previous result. Now consider:

$$A^T E A = (\underline{\alpha}_1, \underline{\alpha}_2, \ldots \underline{\alpha}_N)^T E (\underline{\alpha}_1, \underline{\alpha}_2, \ldots \underline{\alpha}_N)$$

$$= (\underline{\alpha}_1, \underline{\alpha}_2, \ldots \underline{\alpha}_N)^T (\lambda_1 \underline{\alpha}_1, \lambda_2 \underline{\alpha}_2, \ldots \lambda_N \underline{\alpha}_N)$$

$$= \begin{bmatrix} \lambda_1, 0, 0, \ldots \\ 0, \lambda_2, 0, 0, \ldots \\ \\ \\ \ldots\ldots 0, 0, 0, \lambda_N \end{bmatrix}$$

$$= \Lambda$$ as required. ***QED.***

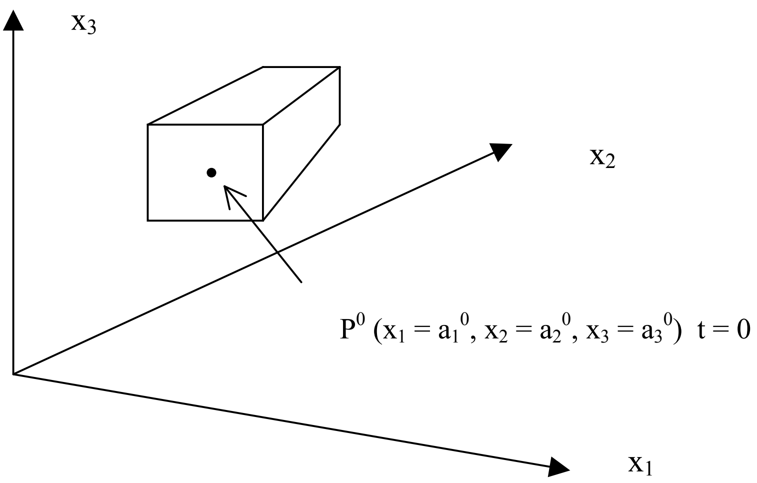
Concept of Continuum

A *continuum* is a macroscopic model of a material: no atomic or quantum effects! The material is infinitely divisible and smoothly varying except, perhaps, for well defined points of discontinuity in the material properties or their gradients. The material may or may not be *homogeneous* – which means having the same properties at all points within the continuum, or within some sub-region of the continuum. It may or may not be *isotropic* – which means having the same properties in all directions at any point. These properties are independent. A plum pudding is isotropic but not homogeneous. A glass-fibre rod is homogeneous but not isotropic.

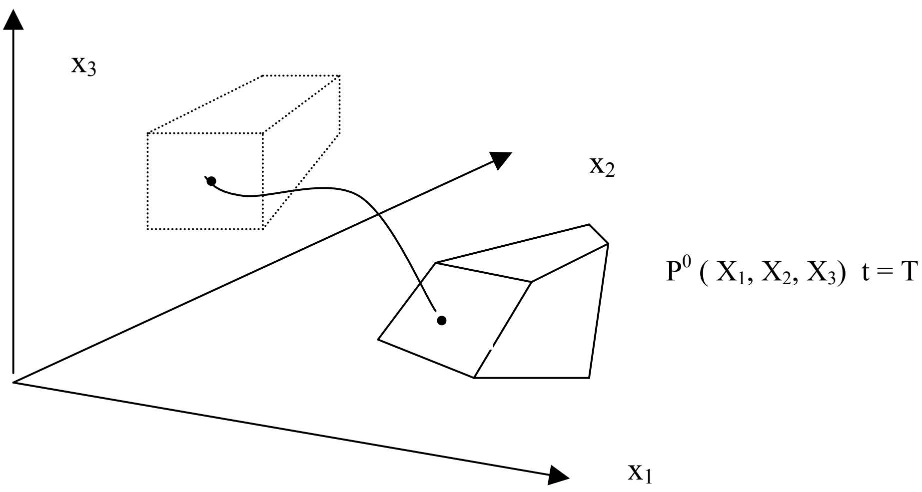
We will often be dealing with *small deformations* of the continuum. Experiment shows that this assumption is usually necessary to invoke the assumption of *elastic behaviour* of the material. However, the methods can be extended (with some care) to finite deformations. And while we will be concerned with solids, much of the theory can be applied to fluids as well.

Eulerian and Lagrangian Coordinates

Consider a cuboid of material within a continuum, centred at P^0 (a_1^0, a_2^0, a_3^0) at t = 0.



After time t = T the continuum has deformed (ie moved, flowed, stretched, rotated) and now the matchbox is centred at (X_1, X_2, X_3)



We can take two views of this.

(1) We can stand at the origin and watch the matchbox move and deform, describing the motion in terms of the (fixed) coordinate system (x_1, x_2, x_3) and t. This is called the Eulerian, or spatial, system of coordinates.

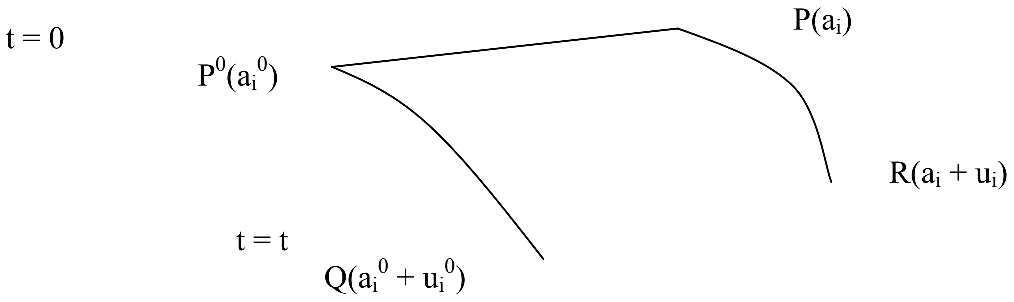
Or (2) We could stand at P^0 and follow the movement of 'our' matchbox, whose motion we would regard as a function of the initial position (a_1^0, a_2^0, a_3^0), t = 0. So we would write X_i = X_i (a_1^0, a_2^0, a_3^0, t), i = 1, 2, 3; $a_i^0 = X_i$ (a_1^0, a_2^0, a_3^0, 0) being the initial condition.

This system of coordinates, depending on the position within the material, is called the Lagrangian, or material, system. Both have their uses.

We are interested not in the bodily movement of the matchbox, but rather with how it deforms with time. So we will ride on the matchbox to observe its changing shape.

The deformation of a continuum

So consider a point P near P^0 (within the matchbox).



In time t, $P^0 \rightarrow Q$, displacement u_i^0, and $P \rightarrow R$, displacement u_i.

We shall consider that u_i^0, u_i are functions of their starting points P^0, P, and t; i.e. we will use Lagrangian coordinates.

Now this is a continuum, so we can assume that all movements, etc are smooth. So expand $u_i(a_1, a_2, a_3)$ as a Taylor Series, viz:

$$\begin{aligned} u_i(a_1, a_2, a_3) &= u_i^0(a_1^0, a_2^0, a_3^0) \\ &+ \partial u_i(a_1, a_2, a_3) / \partial a_1 \,.\, (a_1 - a_1^0) \\ &+ \partial u_i(a_1, a_2, a_3) / \partial a_2 \,.\, (a_2 - a_2^0) \\ &+ \partial u_i(a_1, a_2, a_3) / \partial a_3 \,.\, (a_3 - a_3^0) \\ &+ \text{terms of order } (a_j - a_j^0)^2 \end{aligned}$$

[like $\frac{1}{2} \partial^2 u_i(a_1, a_2, a_3) / \partial a_1 \partial a_2 . (a_1 - a_1^0)(a_2 - a_2^0)$, etc]

for i = 1, 2, 3.

We assume that $(a_j - a_j^0)$ is small, j = 1, 2, 3; so neglect the higher order terms.

Write $\Delta a_j = (a_j - a_j^0)$ = coordinates of P relative to P^0 (for j = 1, 2, 3).

Then to the first order in Δa_j,

$$u_i(a_1, a_2, a_3) - u_i^0(a_1^0, a_2^0, a_3^0) = \sum_{j=1}^{3} (\partial u_i / \partial a_j) \Delta a_j \tag{3}$$

By the summation convention, in place of eq 3 we would write:

$$u_i(a_1, a_2, a_3) - u_i^0(a_1^0, a_2^0, a_3^0) = (\partial u_i / \partial a_j) \Delta a_j \tag{3a}$$

Now $u_i(a_1, a_2, a_3) - u_i^0(a_1^0, a_2^0, a_3^0)$ is the displacement of P relative to P^0; i.e., relative to P^0 we see P move by

$$\Delta u_i = (u_i - u_i^0)$$

So eq 1a now becomes:

$$\Delta u_i = (\partial u_i / \partial a_j)\ \Delta a_j \tag{3b}$$

Note that this is a 'proper' index set equation. Moreover u_i is a vector, so by our earlier result ($\partial u_i / \partial a_j$) is a tensor; and Δu_i and Δa_j are vectors. So all quantities of eqn (3b) are tensors.

E and W

Now we apply the very useful trick of adding and subtracting a convenient amount to ($\partial u_i / \partial a_j$): add and subtract ½ ($\partial u_j / \partial a_i$) -

$$\Delta u_i = (\partial u_i / \partial a_j + \tfrac{1}{2} \partial u_j / \partial a_i - \tfrac{1}{2} \partial u_j / \partial a_i) \Delta a_j$$

divide ($\partial u_i / \partial a_j$) in half and rearrange:

$$= (\tfrac{1}{2} \partial u_i / \partial a_j + \tfrac{1}{2} \partial u_j / \partial a_i + \tfrac{1}{2} \partial u_i / \partial a_j - \tfrac{1}{2} \partial u_j / \partial a_i) \Delta a_j$$

so:

$$\Delta u_i = (\tfrac{1}{2} \partial u_i / \partial a_j + \tfrac{1}{2} \partial u_j / \partial a_i) \Delta a_j + (\tfrac{1}{2} \partial u_i / \partial a_j - \tfrac{1}{2} \partial u_j / \partial a_i) \Delta a_j \quad (4)$$

Now we define the first term in parentheses to the i,j th element of a tensor E; and define the second term to be the i,j th element of a tensor W (they are tensors because they are sums of derivatives of vectors). So we can write eq 4 as a tensor equation (still using the summation convention):

$$\Delta u_i = E_{ij} \Delta a_j + W_{ij} \Delta a_j \quad (4a)$$

(which applies to each component of Δu_i , i = 1, 2, 3.)

If we arrange the components of Δu_i in a column vector $\underline{\mathbf{\Delta u}} = (\Delta u_1, \Delta u_2, \Delta u_3)^T$ then we can write eq4a as an equivalent matrix equation:

$$\underline{\mathbf{\Delta u}} = E\, \underline{\mathbf{\Delta a}} + W\, \underline{\mathbf{\Delta a}} \quad (4b)$$

To recap: eq 4a (or 4b) represents the (small) displacement, relative to P^0 , of points near to P^0, to the first order in Δa_j .

The meaning of W

First note that E and W are symmetric ($E_{ij} = E_{ji}$) and antisymmetric ($W_{ij} = - W_{ji}$) respectively, by construction. This means, for W, that since $W_{ii} = - W_{ii}$ for each i, then $W_{ii} = 0$, ie

$$W = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{bmatrix}$$

Ie W has only three independent components (there is a reason for writing it this way with these signs!)

So define

$$\underline{\boldsymbol{\omega}} = - (W_{23}, W_{31}, W_{12})^T$$

(Note the order! 1st component is W_{23} , rest follow cyclically.)

$\underline{\omega}$ is called the associated vector of W. Now consider

$$W\,\underline{\Delta \mathbf{a}} = \begin{bmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{2,3} \\ W_{31} & -W_{23} & 0 \end{bmatrix} \begin{vmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_3 \end{vmatrix}$$

$$= \begin{bmatrix} W_{12}.\Delta a_2 - W_{31}.\Delta a_3 \\ -W_{12}.\Delta a_1 + W_{23}.\Delta a_3 \\ W_{31}.\Delta a_1 - W_{23}.\Delta a_2 \end{bmatrix}$$

Which looks like a cross product; indeed –

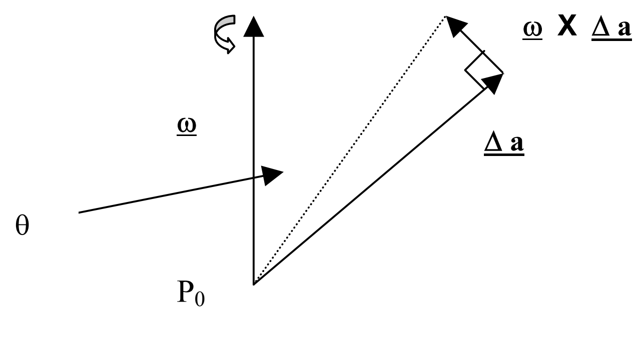
$$W\,\underline{\Delta \mathbf{a}} = \det \begin{vmatrix} \underline{\mathbf{x}}_1 & \underline{\mathbf{x}}_2 & \underline{\mathbf{x}}_3 \\ -W_{23} & -W_{31} & -W_{12} \\ \Delta a_1 & \Delta a_2 & \Delta a_3 \end{vmatrix}$$

$$= \underline{\omega} \ \mathbf{X} \ \underline{\Delta \mathbf{a}}$$

(NB by $\underline{\mathbf{x}}_i$ we mean the unit vector in the direction of the x_i axis.)

So the W effect of the deformation on $\underline{\Delta \mathbf{a}}$ is the same as that produced by a cross product with the associated vector.

We can interpret $\underline{\omega} \ \mathbf{X} \ \underline{\Delta \mathbf{a}}$ easily. Recall that a cross product is a vector perpendicular to both $\underline{\omega}$ and $\underline{\Delta \mathbf{a}}$, and it is *small* by assumption. So $\underline{\omega} \ \mathbf{X} \ \underline{\Delta \mathbf{a}}$ represents a component of $\underline{\Delta \mathbf{u}}$ (the displacement of the vector $\underline{\Delta \mathbf{a}}$) at right angles to $\underline{\Delta \mathbf{a}}$ and $\underline{\omega}$:



From the figure, $\underline{\omega} \ \mathbf{X} \ \underline{\Delta \mathbf{a}}$ represents a rotation of the end of $\underline{\Delta \mathbf{a}}$ about the vector $\underline{\omega}$; at least, in the limit as $t \rightarrow 0$, or $\underline{\Delta \mathbf{u}} \rightarrow 0$.

So we can interpret W $\underline{\Delta \mathbf{a}}$ (which is the same as $\underline{\omega} \ \mathbf{X} \ \underline{\Delta \mathbf{a}}$) as a (rigid) rotation of the continuum, relative to the reference point P^0 as origin, about an axis $\underline{\omega}$ at P^0. The amount of rotation is

$$\phi = |\underline{\omega} \ \mathbf{X} \ \underline{\Delta \mathbf{a}}| / (\sin\theta \, |\underline{\Delta \mathbf{a}}|) = |\underline{\omega}| \text{ (true for } \phi \text{ small).}$$

where θ is the angle between $\underline{\omega}$ and $\underline{\Delta \mathbf{a}}$ .In practical problems, we may or may not have information about the rotation. Eg, if we are interested in the deformation of the Earth's surface, we cannot by conventional terrestrial surveying estimate how much rotation has occurred, unless we have measurements of some quantity relative to an external frame of reference – such as astronomical observations, Global Positioning System, or palaeomagnetic observations that show the rotation relative to the Earth's magnetic pole.

More about rotations

First, we can use the alternating tensor to write the cross product, so:

$$\underline{\boldsymbol{\omega}} \times \underline{\Delta \mathbf{a}} = \varepsilon_{ijk}\, \omega_j\, \Delta a_k$$

Since we can rotate our coordinate system to any orientation we please without 'upsetting' our physical quantities, rotate it so that $\underline{\boldsymbol{\omega}}$ becomes the x_3 axis. Now a rotation of the body about the x_3 axis by ϕ is described by the orthogonal matrix *R*

$$R = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If ϕ is small, then $\sin\phi \sim \phi$, and $\cos\phi \sim 1$ (to the first order in ϕ). So:

$$R^{\phi} \sim \begin{bmatrix} 1 & -\phi & 0 \\ \phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(to the first order in } \phi \text{)}$$

ie a position $\underline{\Delta \mathbf{a}}$ in the body is rotated to $\underline{\Delta \mathbf{a}}'$ given by:

$$\Delta a'_i = R^{\phi}{}_{ij}\, \Delta a_j = \begin{bmatrix} \Delta a_1 - \phi\, \Delta a_2 \\ \Delta a_2 + \phi\, \Delta a_1 \\ \Delta a_3 \end{bmatrix} \qquad (5)$$

From the previous result:

$$\Delta a'_i = \Delta a_i + \phi\, \varepsilon_{ijk}\, (\underline{\mathbf{x}}_3)_j\, \Delta a_k$$

$\underline{\mathbf{x}}_3 = (0, 0, 1)^T$, so this gives:

$$\Delta a'_i = \Delta a_i + \phi\, \varepsilon_{i3k}\, \Delta a_k$$

$$= \delta_{ik}\, \Delta a_k + \phi\, \varepsilon_{3ki}\, \Delta a_k \quad \text{(cyclic permutation of indices of } \phi\, \varepsilon_{ijk} \text{)}$$

$$= (\delta_{ik} + \phi\, \varepsilon_{3ki})\, \Delta a_k$$

which, by inspection, corresponds to eqn 5. We shall want this formulation of rotation in a moment.

The strain tensor E

What kinds of displacement can a continuum undergo? There are only three: *Displacement* of the origin - which we have eliminated by moving with P^0 – *Rotation*, represented by W, and *Distortion* – ie change of shape and/or volume.

Therefore, since E + W described the whole (Lagrangian) displacement, the Distortion must be described by E.

[NB: we could prove that no part of E could contribute to a further rotation, by showing that rotations are only represented by antisymmetric matrices. An exercise for the reader!]

We will call E the *strain* of the continuum. We have already shown that it is a tensor. Remember that E is symmetric.

Strain tensor E in a new coordinate system

Since E is real and symmetric, by the theorem for real symmetric matrices there exists an orthogonal (3 x 3) transformation A which gives

$$A^T E A = \Lambda \qquad (6)$$

where Λ is a diagonal, 3 x 3 matrix whose entries are the eigenvalues of E. By the theorem, E completely determines A. We know from our discussion of transformations in Part 1 that the columns of A = ($\underline{\alpha_1}$, $\underline{\alpha_2}$, $\underline{\alpha_3}$) can be interpreted as a new set of axes.

The tensor transformation corresponding to eqn 6 is written:

$$\Lambda_{ij} = E'_{ij} = a_{ip} a_{iq} E_{pq} \qquad (6a)$$

where $a_{ip} = A_{pi}$.
So an equivalent statement of the theorem, as it applies to any real symmetric second order tensor E, is that we can find a coordinate system, from the eigenvectors of E, in which the tensor is a diagonal tensor. This is clearly a convenient form to work with (three quantities to deal with instead of 6). Moreover, by the Fundamental Principle the properties of the tensor are unaltered by the coordinate system. Therefore we are perfectly at liberty to choose to operate in this convenient coordinate system.

We can thus use the ($\underline{\alpha_1}$, $\underline{\alpha_2}$, $\underline{\alpha_3}$) coordinate system to describe the deformation of the continuum. Make the transformations to the new system:

$$\underline{\mathbf{\Delta a}} \quad \rightarrow \quad A^T \underline{\mathbf{\Delta a}} = \underline{\mathbf{\Delta a}}',$$

$$\underline{\mathbf{\Delta u}} \quad \rightarrow \quad A^T \underline{\mathbf{\Delta u}} = \underline{\mathbf{\Delta u}}'.$$

so that the strain part of the displacement equation:

$$\underline{\mathbf{\Delta u}} \quad = \quad E \; \underline{\mathbf{\Delta a}}$$

becomes:

$$A^T \underline{\mathbf{\Delta u}} \quad = \quad A^T E \; \underline{\mathbf{\Delta a}}$$

$$= \quad A^T E \; (A A^T) \; \underline{\mathbf{\Delta a}} \quad \text{(because } A A^T = I\text{)}$$

$$= \quad (A^T E A)(A^T \underline{\mathbf{\Delta a}})$$

ie $$\underline{\mathbf{\Delta u}'} \quad = \quad \Lambda \; \underline{\mathbf{\Delta a}'}$$

ie $\quad \underline{\Delta \mathbf{u}}' = \begin{bmatrix} \lambda_1, 0, 0 \\ 0, \lambda_2, 0 \\ 0, 0, \lambda_3 \end{bmatrix} \underline{\Delta \mathbf{a}}'$

is the equation that describes the strain deformation of the continuum in this coordinate system. As noted before, we have the simplified circumstances that Λ is a diagonal matrix.

The axes ($\underline{\alpha_1}$, $\underline{\alpha_2}$, $\underline{\alpha_3}$) are called the *Principal Axes* of the deformation and the diagonal elements λ_1, λ_2, and λ_3 are called the *Principal Strains*.

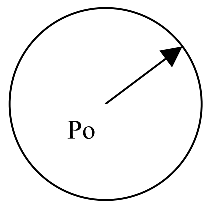
Deformation of a unit sphere

It is easy to analyse the effect of Strain using the Principal Axes system, where (dropping the primes ')

$$\underline{\Delta \mathbf{u}} = \Lambda \, \underline{\Delta \mathbf{a}} \tag{7}$$

$$\Lambda = \begin{bmatrix} \lambda_1, 0, 0 \\ 0, \lambda_2, 0 \\ 0, 0, \lambda_3 \end{bmatrix} \tag{7a}$$

So now consider an imaginary sphere (like our imaginary matchbox), of radius 1, embedded in the continuum at t = 0.



$|\underline{\Delta \mathbf{a}}| = 1$

Any point on the sphere, at $\underline{\Delta \mathbf{a}}$, satisfies $\Delta a_1^2 + \Delta a_2^2 + \Delta a_3^2 = 1$ at t = 0. After deformation (t = T), the point at $\underline{\Delta \mathbf{a}}$ has been moved to $\underline{\Delta \mathbf{a}}' = \underline{\Delta \mathbf{a}} + \underline{\Delta \mathbf{u}}$.

$\underline{\Delta \mathbf{u}}$ is given by eq 1:

$$\underline{\Delta \mathbf{u}} = \Lambda \, \underline{\Delta \mathbf{a}}$$

$$= \begin{bmatrix} \lambda_1 \Delta a_1 \\ \lambda_2 \Delta a_2 \\ \lambda_3 \Delta a_3 \end{bmatrix}$$

So $\quad \underline{\Delta \mathbf{a}}' = \underline{\Delta \mathbf{a}} + \underline{\Delta \mathbf{u}}$.

$$= \begin{bmatrix} (1 + \lambda_1) \Delta a_1 \\ (1 + \lambda_2) \Delta a_2 \\ (1 + \lambda_3) \Delta a_3 \end{bmatrix}$$

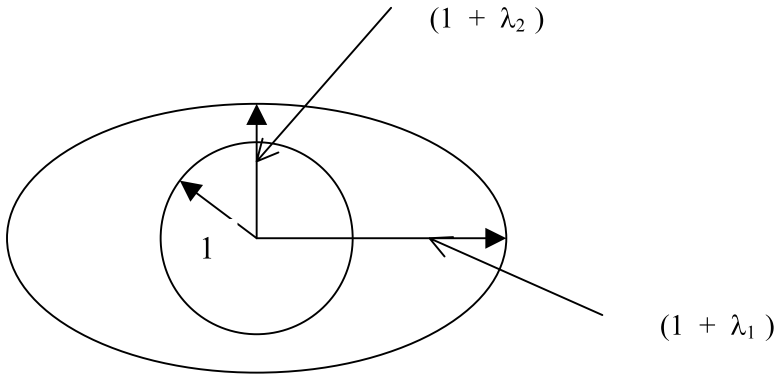
So $\quad \Delta a_1^2 + \Delta a_2^2 + \Delta a_3^2$

$$= [\Delta a_1' / (1 + \lambda_1)]^2 + [\Delta a_2' / (1 + \lambda_2)]^2 + [\Delta a_3' / (1 + \lambda_3)]^2 = 1$$

which is the equation of an *ellipsoid*, whose Principal Axes align with the Principal Axes of Strain, and whose semi-axes are:

$$(1 + \lambda_1), \quad (1 + \lambda_2), \quad (1 + \lambda_3)$$

Suppressing one dimension (and exaggerating the strain, which is *small*):



Several results follow:

(i) If $\lambda_2, \lambda_3 = 0, \lambda_1 \neq 0$, then the only deformation is in the x_1 direction, where a length L is deformed to a length $(1 + \lambda_1) L$, so the fractional change in length (ie the 1-D strain) is:

$$[(1 + \lambda_1) L - L] / L = \lambda_1$$

That is, λ_1 is the strain according to the usual 1-D definition of strain.

Notice that an *extension* is *positive* and a *contraction negative*.

(ii) The fractional change in volume – or the volumetric strain, called the *Dilatation* -

= (volume of ellipsoid – volume of sphere)/(volume of sphere)

$$= \left(4/3\, \pi (1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3) - 4/3\, \pi\, 1^3 \right) / (4/3\, \pi\, 1^3)$$

$$= (1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3) - 1$$

If (as we usually assume) the strains are small compared to 1, then we can ignore terms like $\lambda_1 \lambda_2$ in the expansion of $(1 + \lambda_1)(1 + \lambda_2)(1 + \lambda_3)$. So the Dilatation is approximately:

$$= (1 + \lambda_1 + \lambda_2 + \lambda_3) - 1$$

$$= \lambda_1 + \lambda_2 + \lambda_3$$

which is the *Trace* of Λ.

Invariants of E

We need to go back a bit and consider more of the consequences of the equation 6:

$$A^T E A = \Lambda$$

The columns $\underline{\alpha_i}$ of A are the eigenvectors of E corresponding to eigenvalues λ_i. So we can find the eigenvalues of E, i.e. the Principal Strains, by solving the Characteristic Equation:

$$| E - \lambda I | = 0$$

That is (remembering that E is symmetric!):

$$\begin{vmatrix} E_{11} - \lambda & E_{12} & E_{13} \\ E_{12} & E_{22} - \lambda & E_{23} \\ E_{13} & E_{23} & E_{33} - \lambda \end{vmatrix} = 0$$

ie
$$\begin{aligned}&(E_{11} - \lambda)\,[\,(E_{22} - \lambda)(E_{33} - \lambda) - E_{23}{}^2] \\ &+ E_{12}\,[E_{13}\,E_{23} - E_{12}\,(E_{33} - \lambda)] \\ &+ E_{13}\,[E_{12}\,E_{23} - E_{13}\,(E_{22} - \lambda)] \qquad = 0\end{aligned}$$

ie
$$\begin{aligned}&- \lambda^3 \\ &+ \lambda^2\,[E_{11} + E_{22} + E_{33}] \\ &+ \lambda\,[E_{12}{}^2 + E_{13}{}^2 + E_{23}{}^2 - E_{11}\,E_{33} - E_{11}\,E_{22} - E_{11}\,E_{33}] \\ &+ [E_{11}E_{22}E_{33} + E_{12}\,E_{13}\,E_{23} + E_{13}\,E_{12}\,E_{23} - E_{23}{}^2\,E_{11} - E_{12}{}^2\,E_{33} - E_{13}{}^2\,E_{22}] = 0 \qquad (8)\end{aligned}$$

which is of course a cubic, which will in general have three complex roots. As we have seen, E being symmetric guarantees that the roots are *real*. Note that the coefficient of λ^2 is Trace (E) and that the coefficient of λ^0 (=1) is | E |.

Now since eq 8 has three real roots (call them λ_1, λ_2, λ_3), we can write eq 8 equivalently as:

$$- (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

which is identically equal to eq 8. Compare the multipliers of λ^2 :

$$\lambda_1 + \lambda_2 + \lambda_3 = E_{11} + E_{22} + E_{33}$$

This must be true for *any* coordinate system ie the Trace of E is invariant to changes of coordinates, and is indeed called an *invariant* of E. The multipliers of λ^1 and λ^0 are similarly *invariants* of E.

However, our particular interest is in Trace (E) which we have just proved to be equal to the *dilatation* in every coordinate system.

Deviatoric Strain

Put δ (for dilatation) $= \lambda_1 + \lambda_2 + \lambda_3 = E_{11} + E_{22} + E_{33}$

And subtract $\delta/3$ I from E to make E * :

$$\begin{aligned} E &= (E - \delta/3\,I) + \delta/3\,I \\ &= E^* + \delta/3\,I \end{aligned}$$

E^* is called the *Deviatoric Strain*. Note that Trace (E^*) = Trace (E) – 3 x $\delta/3$ = 0; ie the dilatation of E^* is 0. So E^* describes the deformation of the continuum without volume change.

$$
\begin{aligned}
\text{Now:}\quad A^T E A = \Lambda &= A^T (E^* + \delta/3\, I)\, A \\
&= A^T E^* A + A^T \delta/3\, I\, A \\
&= A^T E^* A + \delta/3\; A^T A \\
&= A^T E^* A + \delta/3\; I
\end{aligned}
$$

$$
\text{So:}\quad A^T E^* A = \Lambda - \delta/3\; I = \begin{bmatrix} \lambda_1 - \delta/3 & 0 & 0 \\ 0 & \lambda_2 - \delta/3 & 0 \\ 0 & 0 & \lambda_3 - \delta/3 \end{bmatrix}
$$

which is diagonal. *So E and E * have the same Principal Axes.*

General equation for the Principal Axes in Plane strain

We shall now consider the components of E in a plane. If there is no deformation in the 3rd direction, this is called Plane Strain. It has engineering applications e.g. in assessing the deformation of sheets of materials, and in the deformation of the Earth.

We assume that there is no deformation in the a_3 direction, nor dependence of strain in any other direction on the a_3 direction. So:

$$
E = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

And we solve:

$$
E\,\underline{\alpha} = \lambda\,\underline{\alpha}, \text{ ie}
$$

$$
\begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \lambda\,\alpha_1 \\ \lambda\,\alpha_2 \\ \lambda\,\alpha_3 \end{bmatrix} \qquad (9)
$$

The last row gives $0 = \lambda\,\alpha_3$. Therefore $\alpha_3 = 0$, since $\lambda \neq 0$. So the Principal Axes with non-trivial $\lambda \neq 0$ lie in the a1 , a2 plane. The third one is perpendicular to them, and must therefore be a_3. So in the case of Plane Strain we can suppress the 3rd row and column of E .

Then the first two rows of eq 4 give:

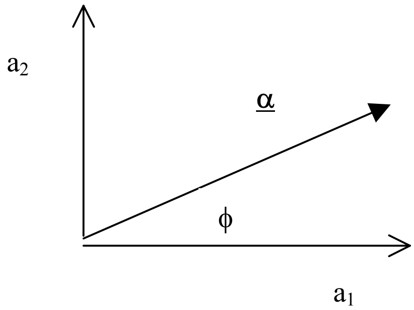
$$
E_{11}\,\alpha_1 + E_{12}\,\alpha_2 = \lambda\,\alpha_1 \qquad (10a)
$$

$$
E_{12}\,\alpha_1 + E_{22}\,\alpha_2 = \lambda\,\alpha_2 \qquad (10b)
$$

Dividing (10a) by α_1 and (10b) by α_2 and equating:

$$
E_{11} + E_{12}\,\alpha_2/\alpha_1 = \lambda = E_{22} + E_{12}\,\alpha_1/\alpha_2 \qquad (11)
$$

Now $\alpha_2 / \alpha_1 = \tan \phi$;



which is the tangent of the angle between the Principal Axis and the a_1 axis, which is what we want. Substitute $\alpha_2 / \alpha_1 = \tan \phi$ in eq 11:

$$E_{11} + E_{12} \tan \phi = E_{22} + E_{12} / \tan \phi \qquad (11a)$$

So: $\tan \phi \, (E_{11} - E_{22}) = E_{12} \, (1 - \tan^2 \phi)$

Therefore:

$$2 \tan \phi \,/\, (1 - \tan^2 \phi) = 2 E_{12} / (E_{11} - E_{22})$$

Now the LHS is $\tan 2\phi$.

So: $$\phi = \tfrac{1}{2} \tan^{-1} \{ 2 E_{12} / (E_{11} - E_{22}) \} \qquad (12)$$

is the angle that (one of) the Principal Axes makes with the a_1 axis. This useful result is worth remembering. Note that $\tan 2(\phi + \pi/2) = \tan(2\phi + \pi) = \tan 2\phi = 2 E_{12} / (E_{11} - E_{22})$

ie $\phi + \pi/2$ is also a solution of eq 12 which, of course, gives the angle of second Principal Axis at $\pi/2$ to the first Principal Axis.

Isotropic Tensors

A material is *isotropic* if its physical properties are the same in any direction. For example, if we measure the extension of a steel plate in response to a force of the same magnitude applied in different directions, we would expect the strain to be the same in each case. We would expect the steel to be isotropic. Glass is isotropic, wood is not.

Isotropy is an important property of materials and fields described by tensors, so we are going to spend a little time characterising isotropic tensors.

Precisely, we say that a tensor is isotropic if its components are unaltered *in value* by (rotational) orthogonal transformations. Like the properties of a steel plate, if we determine the tensor in different orientations, we get the same components.

Isotropic tensors of zero rank.

These are scalars, which are the same in every coordinate system. So every tensor of zero rank is isotropic.

Isotropic tensors of rank 1.

Let vector **v** be isotropic. Since it is a tensor, then

$$v'_i = a_{ij} v_j \qquad (1)$$

for any orthogonal rotation a_{ij}. Since it is isotropic, we require

$$v'_i = v_i$$

for any orthogonal transformation.

Therefore consider a 180 degree rotation about the x_1 axis:

$$a^{180}{}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

From eqn 1,

$$v'_2 = -v_2$$
$$v'_3 = -v_3$$

Hence $v_2 = v_3 = 0$. Similarly for v_1. Hence there are *no (non-trivial) isotropic vectors of rank 1.*

Isotropic tensors of rank 2.

The Kroneker (identity) tensor is isotropic. Proof –

$$\delta'_{ij} = a_{ip} a_{jq} \delta_{pq}$$

$$= a_{ip} a_{jp}$$

$$= \delta_{ij} \qquad \text{because } a_{ij} \text{ is orthogonal.}$$

It can be shown that every second order tensor of rank 2 is of the form $k\,\delta_{ij}$, where k is a scalar.

Outline of proof

Let b_{ij} be a general isotropic tensor of rank 2.

(i) b_{ij} is diagonal. Rotate b_{ij} by 180 degrees about the x_1 axis:

$$b'_{ij} = a^{180}{}_{ip} \; a^{180}{}_{jq} \, b_{pq}$$

so $$b'_{12} = a^{180}{}_{11} \, a^{180}{}_{22} \, b_{12} + \quad \text{zero terms}$$

$$= -b_{12}$$

Since b'_{ij} is isotropic, b_{12} must be zero; similarly for other off-diagonal terms.

(ii) Now consider a small rotation θ of the axes about the x_3 axis. Recall our development of an expression for the rotation of a *body* if θ is small:

$$R^{\theta} \sim \begin{bmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(to the first order in } \theta \text{)}$$

This is a rotation of the *body* through -θ, or the axes through θ. The transformation in tensor notation is

$$R^{\theta}{}_{i\,k} = (\delta_{i\,k} - \theta\, \varepsilon_{3\,k\,i}) = (\delta_{i\,k} + \theta\, \varepsilon_{3\,i\,k})$$

And we have

$$b'_{i\,j} = R^{\theta}{}_{i\,p}\; R^{\theta}{}_{j\,q}\, b_{p\,q}$$

$$= (\delta_{i\,p} + \theta\, \varepsilon_{3\,i\,p})\,(\delta_{j\,q} + \theta\, \varepsilon_{3\,j\,q})\, b_{p\,q}$$

$$= \delta_{i\,p}\delta_{j\,q} b_{p\,q} + \theta\, \varepsilon_{3\,i\,p}\delta_{j\,q} b_{p\,q} + \theta\, \delta_{i\,p}\, \varepsilon_{3\,j\,q} b_{p\,q} + \theta^{2} \varepsilon_{3\,i\,p}\, \varepsilon_{3\,j\,q}\, b_{p\,q}$$

$$= b_{i\,j} + \theta\,(\varepsilon_{3\,i\,p} b_{p\,j} + \varepsilon_{3\,j\,q} b_{i\,q}) \quad \text{(neglecting } \theta^{2}\text{)}$$

$$= b_{i\,j} \text{ since it is isotropic.}$$

Therefore:

$$(\varepsilon_{3\,i\,p} b_{p\,j} + \varepsilon_{3\,j\,q} b_{i\,q}) = 0$$

Take i = 1, j = 2:

$$(\varepsilon_{3\,1\,p} b_{p\,2} + \varepsilon_{3\,2\,q} b_{1\,q}) = 0$$

which has non-zero terms only for p = 2 and q = 1; hence

$$(\varepsilon_{3\,1\,2} b_{2\,2} + \varepsilon_{3\,2\,1} b_{1\,1}) = 0$$

or

$$+ b_{2\,2} - b_{1\,1} = 0$$

or

$$b_{2\,2} = b_{1\,1}$$

Similarly for $b_{3\,3}$. I.e. the diagonal terms are that same and we can write:

$$b_{i\,j} = k\,\delta_{i\,j}$$

as required.

Isotropic tensors of rank 3

The alternating tensor $\varepsilon_{i\,j\,k}$ is isotropic. All other isotropic tensors of rank 3 are multiples of it. The proof is similar to that for isotropic tensors of rank 2 (see Fung 'A first course in continuum mechanics', p140).

Isotropic tensors of rank 4

These will be important to us, because of their implications for the relationship between stress and strain - Hooke's Law - in an isotropic medium.

Since δ_{ij} is isotropic, it is easily shown, using the same procedure as for δ_{ij}, that $\delta_{ij}\delta_{km}$, $\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}$ and $\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk} = \varepsilon_{sij}\varepsilon_{skm}$ are also isotropic.

Furthermore, a general isotropic tensor of rank 4, say u_{ijkm}, can be written in the form:

$$u_{ijkm} = \lambda\,\delta_{ij}\delta_{km} + \mu\,(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) + \nu\,(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk})$$

Furthermore, if u_{ijkm} has symmetry properties: $u_{ijkm} = u_{jikm}$ and $u_{ijkm} = u_{ijmk}$,
Then

$$\boldsymbol{u_{ijkm} = \lambda\,\delta_{ij}\delta_{km} + \mu\,(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})}$$

is the general form of an isotropic, symmetric tensor of rank 4, where λ *and* μ *are arbitrary constants.*

(We will not prove this. The argument follows along the lines of the proof for tensors of rank 2 – see Fung 'A first course in continuum mechanics', p141.)

Bringing together Stress and Strain - Hooke's Law

We are now able to properly consider Hooke's law in its most general form, which can be stated as: "Stress (as a tensor) is linearly related to strain (as a tensor)". That is -

$$\mathbf{S}_{ij} = \mathbf{C}_{ijkl}\,\mathbf{E}_{kl} \quad \textit{(summation over k, l)}$$

where the $3^4 = 81$ coefficients C_{ijkl} are independent of E and S, but may depend on location in the medium (so they may not be "constants").

A material obeying Hooke's Law is called *elastic*. Hooke's Law applies quite well to real materials when the strains are small.

C_{ijkl} is a 4^{th} order tensor i.e. it transforms according to:

$$C_{ijkl}' = a_{ip}\,a_{jq}\,a_{kr}\,a_{ls}\,C_{pqrs}$$

We will not prove this, but it follows from the definition of C_{ijkl} and that E and S are both tensors.

Reducing the number of coefficients C_{ijkl}

(i) Symmetry of S_{ij} and E_{kl}

Since S_{ij} is symmetric, $S_{ij} = S_{ji}$,

then: $C_{ijkl}\,E_{kl} = C_{jikl}\,E_{kl}$

and since E_{kl} is symmetric,

then: $C_{ijkl}\,E_{kl} = C_{ijkl}\,E_{lk} =$ (renaming) $C_{ijlk}\,E_{kl}$

so there are only really 6 independent ' i, j ' parts of C and only 6 independent 'k , l ' parts. So we have reduced the number of coefficients C_{ijkl} to 6 x 6 = 36. This is a completely general result, resulting from the symmetry of stress and strain.

(ii) In an Isotropic medium, the Principal Axes of E and S coincide. This is important: it means that we can infer the Principal Axes of Stress from measurements of the strain tensor, which are often much easier to make.

Proof: Choose the axes to be the Principal Strain Axes, so that E_{kl} is diagonal ($E_{kl} = 0, k \neq l$).

Then: $S_{ij} = C_{ijkl} E_{kl} = C_{ij11} E_{11} + C_{ij22} E_{22} + C_{ij33} E_{33}$

(other terms in the summation are zero).

Now rotate the axes through 180 degrees about the x_3 axis. This is achieved with the transformation:

$$a^{180}{}_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ie $a_{ip} = \pm 1$ for i = p, $a_{ip} = 0$ otherwise.

We now invoke the assumption of isotropy and require that C_{ijkl} is unchanged by this rotation ie the relationship between components of stress and strain is the same whether we take the + **x** or – **x** direction, etc. So:

$$C_{ijkl}' = C_{ijkl}$$

So in the rotated coordinate system:

$$S_{ij}' = C_{ijkl}' E_{kl}' = C_{ijkl} E_{kl}'$$

and:

$$E_{kl}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix}$$

$$= E_{kl}$$

So:

$$S_{ij}' = C_{ijkl} E_{kl} = S_{ij},$$

ie S_{ij} is unchanged by the rotation.

But:

$$S_{ij}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_{11} & S_{12} & -S_{13} \\ S_{21} & S_{22} & -S_{23} \\ -S_{31} & -S_{32} & S_{33} \end{bmatrix}$$

So: $-S_{31} = S_{31} \Rightarrow S_{31} = 0$, and $-S_{32} = S_{32} \Rightarrow S_{32} = 0$,

And rotation through 180 degrees about another axis would give $S_{12} = 0$ as well. So we have that S_{ij} is diagonal ie it is in its Principal Axis form, like E. QED.

(iii) *Form of C_{ijkl} and Hooke's law for an Isotropic medium*

For an isotropic medium, and because of the symmetry of S and E, we have that C_{ijkm} can be written with complete generality as:

$$C_{ijkm} = \lambda \delta_{ij}\delta_{km} + \mu (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$$

Thus

$$S_{ij} = \{\lambda \delta_{ij}\delta_{km} + \mu (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) \} E_{km}$$

$$= \lambda \delta_{ij}\delta_{km} E_{km} + \mu (\delta_{ik}\delta_{jm} E_{km} + \delta_{im}\delta_{jk} E_{km})$$

Since δ_{ij} is the identity,

$$S_{ij} = \lambda \delta_{ij} E_{kk} + \mu (E_{ij} + E_{ij})$$

i.e.

$$\boldsymbol{S_{ij} = \lambda \delta_{ij} E_{kk} + 2 \mu E_{ij}} \qquad \textbf{(2)}$$

where E_{kk} is the dilatation = $E_{11} + E_{22} + E_{33}$

This then is the general form of Hooke's law for isotropic materials. It has just two parameters – the *Lamé constants* λ and μ (remember – they could depend on position within the material).

The ratio of any stress component to a corresponding strain component is called an *elastic modulus*.

e.g. $S_{12} = 2 \mu E_{12} \quad (\delta_{12} = 0)$

so: $S_{12} / E_{12} = 2 \mu$; μ is called the *Shear Modulus*.

NB the '2' arises historically from the definition of E_{ij}, which has the ½ ; viz:

$$E_{ij} = ½ (\partial u_i / \partial x_j + \partial u_j / \partial x_i)$$

So: $S_{ij} = \mu (\partial u_i / \partial x_j + \partial u_j / \partial x_i) + \lambda \partial u_k / \partial x_k \delta_{ij}$

e.g. 2

$$S_{ii} = \lambda \delta_{ii} E_{kk} + 2 \mu E_{ii}$$

$$= \text{(re-labeling } E_{ii}\text{)} \ \lambda 3 E_{kk} + 2 \mu E_{kk}$$

$$= (3 \lambda + 2 \mu) E_{kk}$$

S_{ii} is the sum of the diagonal elements of S, and is analogous to the dilatation. The pressure p is defined to be –

$$p = - 1/3 \, S_{ii} \text{ (= - mean normal stress)}$$

(Remember tensions are positive, compressions negative).

So the ratio: $-p/E_{kk}$ is that ratio of (-)pressure to volumetric change = ($\lambda + 2/3 \mu$). This is called the *Bulk Modulus,* often denoted by κ.

e.g. 3 *Uniaxial extension* occurs when $S_{11} \neq 0$ and $S_{ij} = 0$ for $i, j \neq 1, 1$.

The ratio S_{11} / E_{11} in uniaxial extension is called Young's Modulus (see assignment).

Newtonian Fluid

A *Newtonian fluid* is a viscous fluid in which the shear stress is linearly proportional to the *rate* of deformation. It is a useful model for many applications, for stiff fluids e.g. the Earth's mantle.

First, in place of the strain tensor E_{ij} we define a rate of strain tensor V_{ij} where we have replaced displacements u_i in the definition of E_{ij} by velocities v_i:

$$V_{ij} = \tfrac{1}{2} (\partial v_i / \partial x_j + \partial v_j / \partial x_i)$$

(think of the displacements in the derivation of E_{ij} occurring in unit time).

Then the full constitutive relationship between stress and rate of strain for a Newtonian Fluid is:

$$S_{ij} = - p \delta_{ij} + D_{ijkl} V_{kl}$$

Where p is, again, the pressure, and we have a set of constants D_{ijkl} in place of C_{ijkl} in Hooke's law. For an isotropic fluid, this reduces (similarly to an isotropic elastic solid), to:

$$S_{ij} = - p \delta_{ij} + \lambda \delta_{ij} V_{kk} + 2 \mu V_{ij}$$

Contracting this gives:

$$S_{ii} = - p \delta_{ii} + \lambda \delta_{ij} V_{kk} + 2 \mu V_{ii}$$

$$= - 3 p + (3 \lambda + 2 \mu) V_{kk}$$

So the identification of $p = - 1/3 \, S_{ii}$ is equivalent to requiring

$$(3 \lambda + 2 \mu) = 0 \quad \text{or} \quad \lambda = - 2/3 \mu$$

which implies that the *rate* of dilatation is not affected by the pressure.

So we have:

$$S_{ij} = -p\,\delta_{ij} + 2\,\mu\,V_{ij} - 2/3\,\mu\,\delta_{ij}\,V_{kk} \qquad (3)$$

A fluid obeying eqn(3) is called a *Stokes fluid* after the 19th applied mathematician George Stokes. μ is called the *viscosity*. If in eqn(2) $\mu = 0$, we have a *non-viscous fluid*, with constitutive equation:

$$S_{ij} = -p\,\delta_{ij} \qquad (4)$$

An Introduction to Tensor Calculus

We have already met the derivatives of tensors, and shown that the new entity that results from differentiating a tensor X_{ij} term by term; e.g.

$$\partial X_{ij} / \partial x_k$$

is a tensor (in this case of rank 3). And we have identified the derivatives from vector calculus -

Gradient of a scalar φ – tensor of rank 1: grad $\varphi = \partial \varphi / \partial x_k$

Divergence of a vector v_k – scalar: $\partial v_k / \partial x_k$

Curl of a vector v_k – vector: $= \varepsilon_{ijk}\, \partial v_k / \partial x_j$

Integrals of tensors

In a similar way we can identify various integrals of tensors (illustrated with tensors of rank 2) e.g.

Line integrals $\int_a^b X_{ij}\, d\,l$

Area integrals $\iint_S X_{ij}\, d\,S$

Volume integrals $\iiint_V X_{ij}\, d\,V$

There may be contractions. E.g. if $X_{ij} = v_i\, n_j$, where n_j is the normal to a surface S, then

$$\iint_S v_i\, n_i\, d\,S$$

is the (scalar) flux of v_i through S.

It may be that the integrating variable is a tensor e.g

$$W = \int_0^{E_{ij}} S_{kl}\, dE_{kl}$$

(W is the strain potential energy per unit volume of an elastic material).

This is understood to be the sum of 3 x 3 = 9 separate integrals:

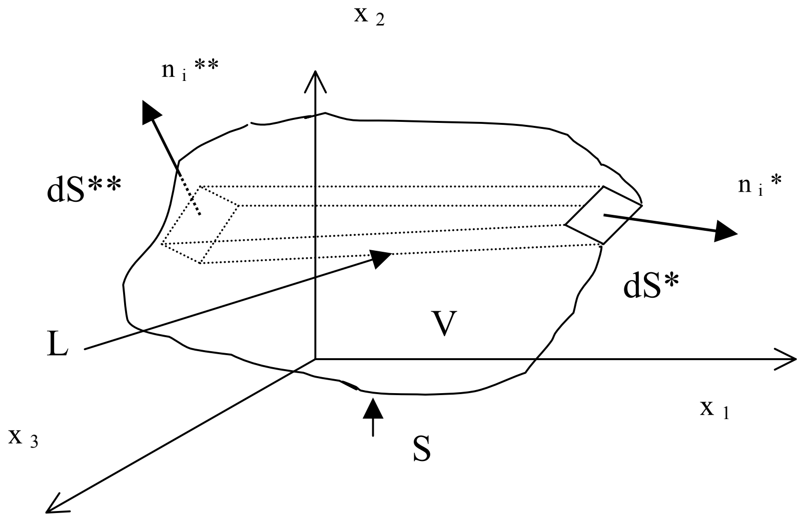
$$W = \sum_{k=1}^{3} \sum_{l=1}^{3} \int_0^{E_{ij}} S_{kl}\, dE_{kl}$$

Gauss's Theorem

Gauss's theorem is one of the most useful theorems in applied mathematics. We will derive a more general result than normally presented.

Consider a convex region V (i.e. no re-entrants or holes) bounded by a surface S. (A non-convex surface can usually be split up into a finite number of convex ones).

Let $A(x_1, x_2, x_3)$ be continuously differentiable in V.



Consider the volume integral:

$$\iiint_V \partial A(x_1, x_2, x_3)/\partial x_1 \; dx_1 dx_2 dx_3$$

Integrate this along the line segment L drawn above (cross section area $dx_2\, dx_3$):

$$\iiint_V \partial A(x_1, x_2, x_3)/\partial x_1 \; dx_1 dx_2 dx_3 = \iint_S (A^* - A^{**})\, dx_2\, dx_3$$

Where A* and A** are the values of A at the ends of the tube, and S is the area of all the ends of the tubes across V. Let the areas of the ends of one tube be dS* and dS**

Now dS* and dS** are the projections of $dx_2\, dx_3$ onto the ends of the tube. If the normals at the ends are n_i*and n_i** , then dS* is the projection of $dx_2\, dx_3$ in the (1, 0, 0) direction, so

$$dx_2\, dx_3 = dS^* \cos(\text{angle between } n_i^* \text{ and } (1, 0, 0) = dS^*\, n_1^*;$$

and dS** is the projection of $dx_2\, dx_3$ in the (-1, 0, 0) direction, so

$$dx_2\, dx_3 = dS^{**} \cos(\text{angle between } n_i^{**} \text{ and } (-1, 0, 0) = -\, dS^{**}\, n_1^{**}.$$

So

$$\iint_S (A^*\, dx_2\, dx_3 - A^{**}\, dx_2\, dx_3 = \iint_S (A^*\, dS^*\, n_1^* + A^{**}\, dS^{**}\, n_1^{**} \qquad (5)$$

The *'s simply mark particular ends of tubes. As we move over S we can write the RHS of eqn(5) as

$$\iint_S A\, n_1\, dS$$

i.e.

$$\iiint_V \partial A/\partial x_1\, dV = \iint_S A\, n_1\, dS$$

Similarly we can calculate $\partial A/\partial x_2$ and $\partial A/\partial x_3$ and get:

$$\iiint_V \partial A/\partial x_i\, dV = \iint_S A\, n_i\, dS$$

Now replace A with an arbitrary, continuously differentiable tensor $X_{ij\ldots n}$
By the same argument we have:

$$\iiint_V \partial X_{ij\ldots n}/\partial x_k\, dV = \iint_S X_{ij\ldots n}\, n_k\, dS$$

This is the most general form of *Gauss's Theorem.*

E.g. 1 let the tensor be a vector v_k Then:

$$\iiint_V \partial v_k/\partial x_k\, dV = \iint_S v_k n_k\, dS$$

which is the familiar "Gauss's Flux Law":

$$\iiint_V \mathrm{div}\, \underline{\mathbf{v}}\, dV = \iint_S \underline{\mathbf{v}} \bullet \underline{\mathbf{n}}\, dS$$

Equations of motion of a continuum

Equation of continuity (Conservation of mass)

Our first application of Gauss's Law is the important Equation of Continuity for a continuum, which is equivalent to a statement that mass is conserved.

Consider a fixed volume of space τ, with matter of (varying) density $\rho(\underline{\mathbf{x}})$. The mass inside τ at t = 0 is

$$M = \iiint_\tau \rho(\underline{\mathbf{x}})\, dx_1\, dx_2\, dx_3$$

The rate of increase of mass in τ is

$$dM/dt = d\left(\iiint_\tau \rho(\underline{\mathbf{x}})\, dx_1\, dx_2\, dx_3\right)/dt$$

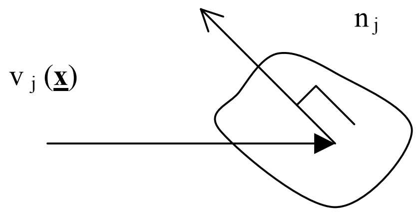
$$= \iiint_\tau \partial\, \rho(\underline{\mathbf{x}})/\partial t \quad dx_1\, dx_2\, dx_3$$

(the rate of change at each point $\underline{\mathbf{x}}$; τ fixed).

Mass is conserved, so this change must equal the mass inflow through the surface S of τ:

$$= - \iint_S \rho(\underline{\mathbf{x}})\, v_j(\underline{\mathbf{x}})\, n_j \, dS$$

where $v_j(\underline{\mathbf{x}})$ is the velocity of the flow (NB only the component normal to S flows in or out, hence the term $v_j(\underline{\mathbf{x}})\, n_j$; and we are interested in *inflow*, hence the minus sign).



By Gauss's Theorem, this flux is:

$$- \iiint_\tau \partial \{ \rho(\underline{\mathbf{x}})\, v_j(\underline{\mathbf{x}}) \} / \partial x_j \; dx_1\, dx_2\, dx_3$$

Hence:

$$\iiint_\tau \partial \rho(\underline{\mathbf{x}}) / \partial t \;\; dx_1\, dx_2\, dx_3 \; + \; \iiint_\tau \partial (\rho(\underline{\mathbf{x}})\, v_j(\underline{\mathbf{x}})) / \partial x_j \;\; dx_1\, dx_2\, dx_3 = 0$$

or

$$\iiint_\tau \{ \partial \rho(\underline{\mathbf{x}}) / \partial t + \partial (\rho(\underline{\mathbf{x}})\, v_j(\underline{\mathbf{x}})) / \partial x_j \} \, dx_1\, dx_2\, dx_3 = 0$$

for *any* volume τ. So the expression in the { } must be zero everywhere in τ . I.e.

$$\partial \rho(\underline{\mathbf{x}}) / \partial t + \partial (\rho(\underline{\mathbf{x}})\, v_j(\underline{\mathbf{x}})) / \partial x_j = 0$$

This is the *Equation of Continuity*. Remember: this is a re-statement of the conservation of mass.

We can differentiate the second term and get the equivalent expression:

$$\partial \rho(\underline{\mathbf{x}}) / \partial t + \partial \rho(\underline{\mathbf{x}}) / \partial x_j \; v_j(\underline{\mathbf{x}}) + \rho(\underline{\mathbf{x}}) \; \partial v_j(\underline{\mathbf{x}}) / \partial x_j = 0$$

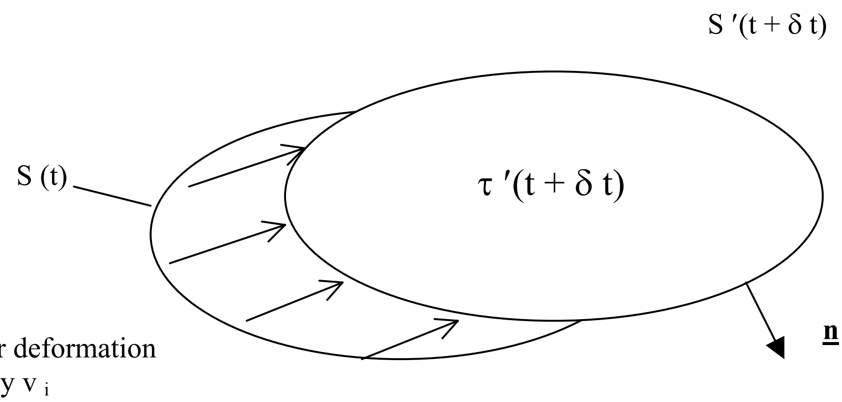
which reduces to $\partial v_j(\underline{\mathbf{x}}) / \partial x_j = 0$ for incompressible (ρ unchanging) fluids.

Extension to moving volume

Consider now the problem of a volume τ *moving with the continuum*. For any quantity $X(\underline{\mathbf{x}}, t)$ we want to be able to compute the *total* rate of change:

$$I = \frac{d}{dt} \left\{ \iiint_{\tau} X(\underline{\mathbf{x}}, t) \, dx_1 \, dx_2 \, dx_3 \right\}$$

where we allow τ to change with time.



We calculate d / dt from first principles:

$$I = \lim_{\delta t \to 0} \left\{ (1/\delta t) \left(\iiint_{\tau'} X(\underline{\mathbf{x}}, t + \delta t) \, dx_1 dx_2 dx_3 - \iiint_{\tau} X(\underline{\mathbf{x}}, t) \, dx_1 dx_2 dx_3 \right) \right\}$$

(NB we are using fixed, or Eulerian coordinates).

Write $\tau' = \tau + \delta\tau$. Then:

$$I = \lim_{\delta t \to 0} \left\{ (1/\delta t) \left(\iiint_{\tau} X(\underline{\mathbf{x}}, t + \delta t) \, dx_1 dx_2 dx_3 - \iiint_{\tau} X(\underline{\mathbf{x}}, t) \, dx_1 dx_2 dx_3 + \iiint_{\delta\tau} X(\underline{\mathbf{x}}, t + \delta t) \, dx_1 dx_2 dx_3 \right) \right\}$$

The first two terms are:

$$I1 = \lim_{\delta t \to 0} \left\{ (1/\delta t) \left(\iiint_{\tau} [X(\underline{\mathbf{x}}, t + \delta t) - X(\underline{\mathbf{x}}, t)] \, dx_1 dx_2 dx_3 \right) \right\}$$

which (we hope, for 'well behaved' X) will converge to

$$I1 = \iiint_{\tau} \partial X(\underline{\mathbf{x}}, t) / \partial t \, dx_1 dx_2 dx_3$$

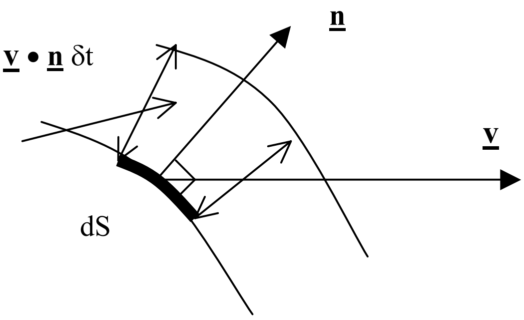
i.e. take the limit inside the integral.

The remaining term is:

$$I2 = \lim_{\delta t \to 0} (1/\delta t) \iiint_{\delta\tau} X(\underline{\mathbf{x}}, t + \delta t) \, dx_1 dx_2 dx_3$$

Now consider an element dS of S.

$dx_1dx_2dx_3 = dS\ \underline{\mathbf{n}} \bullet \underline{\mathbf{v}}\ \delta t$



The volume $dx_1dx_2dx_3$ swept out by dS in δt is given by $dS\ \underline{\mathbf{n}} \bullet \underline{\mathbf{v}}\ \delta t$. Assume that $dx_1dx_2dx_3$ is so small that $\partial X / \partial x_i \ll \partial X / \partial t$ in $\delta\tau$. So take the spatial variation of X to be zero across $dx_1dx_2dx_3$ (but let it vary with dS). Therefore an element of the integral I2 is:

$$X(\underline{\mathbf{x}}, t + \delta t)\ dx_1dx_2dx_3 = X(\underline{\mathbf{x}}, t + \delta t)\ \underline{\mathbf{n}} \bullet \underline{\mathbf{v}}\ dS\ \delta t$$

Thus:

$$\begin{aligned} I2 &= \lim_{\delta t \to 0} (1/\delta t) \iint_S X(\underline{\mathbf{x}}, t + \delta t)\ \underline{\mathbf{n}} \bullet \underline{\mathbf{v}}\ dS\ \delta t \\ &= \lim_{\delta t \to 0} \iint_S X(\underline{\mathbf{x}}, t + \delta t)\ \underline{\mathbf{n}} \bullet \underline{\mathbf{v}}\ dS \\ &= \iint_S \{ X(\underline{\mathbf{x}}, t)\ \underline{\mathbf{v}} \} \bullet \underline{\mathbf{n}}\ dS \\ &= \text{(Gauss)} \iiint_\tau \partial (X(\underline{\mathbf{x}}, t)\ v_i) / \partial x_i\ d\tau \end{aligned}$$

So, putting it all together:

$$d/dt \iiint_\tau X(\underline{\mathbf{x}}, t)\ d\tau = \iiint_\tau \partial X(\underline{\mathbf{x}}, t) / \partial t\ d\tau + \iiint_\tau \partial / \partial x_i (X(\underline{\mathbf{x}}, t)\ v_i)\ d\tau$$

$$= \iiint_\tau \{ \partial X / \partial t + \partial / \partial x_i (X\ v_i) \}\ d\tau \qquad (6)$$

which is the result we were seeking.

Equations of motion of a continuum

Now suppose we have body forces G_i /unit mass inside τ and surface stress forces T_i per unit area i.e. $T_i = S_{ij}\, n_j$ per unit area on the surface A of τ. The total force F_i on τ is therefore:

$$F_i = \iiint_\tau \rho\, G_i\, d\tau + \iint_A S_{ij}\, n_j\ dA$$

Apply Gauss's Theorem to each component of T_i :

$$F_i = \iiint_\tau \rho G_i d\tau + \iiint_\tau \partial S_{ij} / \partial x_j \, d\tau$$

$$= \iiint_\tau \{ \rho G_i d\tau + \partial S_{ij} / \partial x_j \} \, d\tau$$

This is the total force on τ ; so it must equal the rate of change of momentum of τ, given by:

$$d/dt \iiint_\tau (\rho v_i) \, d\tau$$

($\rho d\tau$ = mass; times velocity v_i).

Now apply equation 6 to each component of momentum $X_i = \rho v_i$:

$$\iiint_\tau \{ \rho G_i d\tau + \partial / \partial x_j \, S_{ij} \} \, d\tau = d/dt \iiint_\tau (\rho v_i) \, d\tau$$

$$= \iiint_\tau \{ \partial \rho v_i / \partial t + \partial (\rho v_i v_j / \partial x_j) \} \, d\tau$$

Or: $$\iiint_\tau \{ \rho G_i + \partial / \partial x_j \, S_{ij} - [\partial \rho v_i / \partial t + \partial (\rho v_i v_j) / \partial x_j] \} \, d\tau = 0$$

This applies to *all* volumes τ of the continuum. So the integrand must vanish everywhere:

$$\rho G_i + \partial S_{ij} / \partial x_j - \partial \rho v_i / \partial t - \partial (\rho v_i v_j) / \partial x_j = 0$$

Expand the derivatives:

$$\rho G_i + \partial S_{ij} / \partial x_j - \rho \, \partial v_i / \partial t - v_i \, \partial \rho / \partial t - \partial (\rho v_i v_j) / \partial x_j = 0$$

But $$v_i \, \partial \rho / \partial t + \partial (\rho v_i v_j) / \partial x_j = v_i \, \partial \rho / \partial t + v_i \, \partial (\rho v_j) / \partial x_j + \rho v_j \partial v_i / \partial x_j$$

$$= v_i \, \{ \partial \rho / \partial t + \partial / \partial x_j (\rho v_j) \} + \rho v_j \partial v_i / \partial x_j$$

And the term in { } is zero by the continuity equation. So the equation of motion becomes:

$$\rho G_i + \partial S_{ij} / \partial x_j - \rho \, \partial v_i / \partial t - \rho v_j \partial v_i / \partial x_j = 0$$

Now the acceleration α_i at a point is given by:

$$\alpha_i = d v_i (\underline{\mathbf{x}}, t) / dt = \partial v_i / \partial t + \partial v_i / \partial x_j \, . \, \partial x_j / \partial t$$

$$= \partial v_i / \partial t + v_j \partial v_i / \partial x_j$$

So we have:

$$\rho G_i + \partial / \partial x_j \, S_{ij} - \rho \, \alpha_i = 0$$

Or:

$$\rho \alpha_i = \rho G_i + \partial S_{ij} / \partial x_j \qquad \text{(Euler)}$$

Which is the 'celebrated' Eulerian equation of motion, telling us that the acceleration at a point in a continuum is due to the sum of the body forces plus the *spatial rate of change* of the stress forces.

Navier's equation

We now combine Hooke's Law for isotropic materials:

$$S_{ij} = 2 \mu E_{ij} + \lambda E_{kk} \delta_{ij}$$

with the equation of motion:

$$\rho \alpha_i = \rho G_i + \partial S_{ij} / \partial x_j$$

to obtain the equation of motion for elastic materials. We shall assume that μ and λ are constant (locally). Differentiating Hooke's Law gives:

$$\partial S_{ij} / \partial x_j = 2 \mu \partial E_{ij} / \partial x_j + \lambda \partial E_{kk} / \partial x_j \; \delta_{ij}$$

Now

$$E_{ij} = \tfrac{1}{2} \partial u_i / \partial x_j + \tfrac{1}{2} \partial u_j / \partial x_i$$

And

$$E_{kk} = \partial u_k / \partial x_k$$

So we have (remembering the summation convention):

$$\begin{aligned}
\partial S_{ij} / \partial x_j &= \mu (\partial^2 u_i / \partial x_j \partial x_j + \partial^2 u_j / \partial x_i \partial x_j) + \lambda \partial^2 u_k / \partial x_k \partial x_j \; \delta_{ij} \\
&= \mu (\partial^2 u_i / \partial x_j \partial x_j + \partial^2 u_j / \partial x_i \partial x_j) + \lambda \partial^2 u_k / \partial x_k \partial x_i \\
\text{(re-gathering)} \quad &= \mu \partial^2 u_i / \partial x_j \partial x_j + (\mu + \lambda) \; \partial^2 u_k / \partial x_k \partial x_i
\end{aligned}$$

So Euler's equation:

$$\rho \alpha_i = \rho G_i + \partial S_{ij} / \partial x_j$$

gives

$$\rho \alpha_i = \rho G_i + \mu \partial^2 u_i / \partial x_j \partial x_j + (\mu + \lambda) \; \partial^2 u_k / \partial x_k \partial x_i \qquad (7)$$

NB $\partial^2 / \partial x_j \partial x_j \equiv \nabla^2$, and $\partial^2 u_k / \partial x_k \partial x_i = \nabla \nabla \bullet \underline{\mathbf{u}}$; so we can write eqn(7) in vector notation as:

$$\rho \underline{\alpha} = \rho \underline{\mathbf{G}} + \mu \nabla^2 \underline{\mathbf{u}} + (\mu + \lambda) \; \nabla \nabla \bullet \underline{\mathbf{u}} \qquad (7^*)$$

Either way, eqn (7) is *Navier's Equation*.

We now assume that body forces are negligible (the principal one in practice is often gravity) and if we consider:

$$\alpha_i = d^2 u_i(\underline{\mathbf{x}}, t) / dt^2$$

$$= d(\partial u_i / \partial t + \partial u_i / \partial x_j . d x_j / dt) / dt$$

$$= d(\partial u_i / \partial t + 0) / dt \qquad \text{(because the } x_j \text{ are fixed),}$$

$$= \partial^2 u_i / \partial t^2 + \partial (\partial u_i / \partial t) / \partial x_j . d x_j$$

$$= \partial^2 u_i / \partial t^2$$

for the same reason.

So we get:

$$\boxed{\rho\, \partial^2 u_i / \partial t^2 = \mu\, \partial^2 u_i / \partial x_j \partial x_j + (\mu + \lambda)\, \partial^2 u_k / \partial x_k \partial x_i}$$

(Navier's equation without body forces)

Navier-Stokes equation for fluid flow

In place of Hooke's law, we apply Euler's equation to the constituent equation for fluids:

$$S_{ij} = -p\, \delta_{ij} + \lambda\, \delta_{ij} (\partial v_k / \partial x_k + \mu (\partial v_i / \partial x_j + \partial v_j / \partial x_i)$$

So:

$$\partial S_{ij} / \partial x_j$$

$$= -\partial p / \partial x_j\, \delta_{ij} + \mu (\partial^2 v_i / \partial x_j \partial x_j + \partial^2 v_j / \partial x_i \partial x_j) + \lambda\, \partial^2 v_k / \partial x_k \partial x_j\ \delta_{ij}$$

$$= -\partial p / \partial x_i + \mu\, \partial^2 v_i / \partial x_j \partial x_j + (\lambda + \mu)\, \partial^2 v_k / \partial x_k \partial x_i$$

(for μ and λ constant). And Euler's equation gives:

$$\rho\, \alpha_i = \rho\, G_i - \partial p / \partial x_i + \mu\, \partial^2 v_i / \partial x_j \partial x_j + (\lambda + \mu)\, \partial^2 v_k / \partial x_k \partial x_i$$

These are the Navier-Stokes equations for constant μ and λ . The motion must also satisfy the continuity equation:

$$\partial \rho / \partial t + \partial (\rho\, v_j) / \partial x_j = 0$$

These equations cover a huge range of fluid flows, from atmospheric circulations, through water currents, eddies and waves, to slow flows of treacly fluids. They are in general very difficult to solve. e.g for steady flow ($\alpha_i = 0$) in an incompressible fluid ($\partial v_k / \partial x_k = 0$),

$$\rho\, G_i - \partial p / \partial x_i + \mu\, \partial^2 v_i / \partial x_j \partial x_j = 0$$

the third term is the Laplacian $\nabla^2 v_i$ .The flow is driven by body forces G_i and the pressure gradient $\partial p / \partial x_i$.