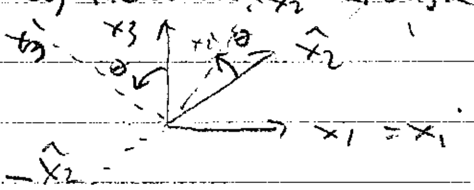
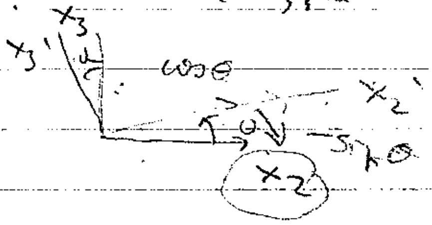
Tutorial Two and Three

Do Tutorial One Questions 4A, 5 + 6

(1) Construct Transformation matrices A for giving the coordinates of a vector $\vec{p}$ in a new coordinate system, using the convention $\vec{p}(new) = A^T p(old)$, for:

a) Rotation through $\theta°$ about x_1 axis: in $x_2 \to x_3$ plane





In both systems: $x_1 = x_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha_1$

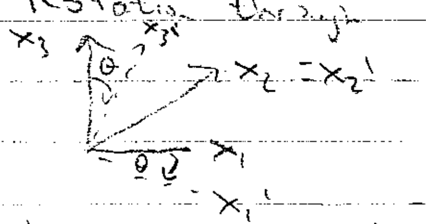
$$x_2' = \alpha_2 = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} \qquad x_3' = \alpha_3 = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$ = standard eqn seen before.

$\vec{p}_{new} = A^T p(old)$ e.g. let $p(old) = \hat{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Then $A^T p(old) = \begin{pmatrix} 0 \\ \cos\theta \\ -\sin\theta \end{pmatrix}$ ✓ $\longrightarrow$ x_2 in new system but x_2'

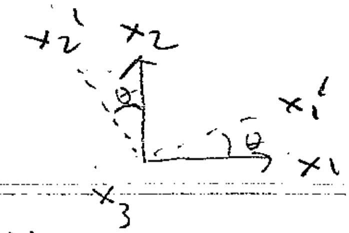
(b) Rotation through $\theta°$ about x_2 axis:



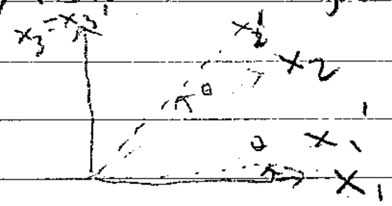
$$A = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \longleftrightarrow A^T = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$x_1' = \begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix} \qquad x_2' = x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad x_3' = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

Test: on x_1 axis $\vec{p}_{new} = A^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix}$ $\rightarrow$ this is x_1 as expressed in the new system



1 c) Rotation through $\theta°$ about x_3 axis:



$x_1' = a_1 = \begin{pmatrix} \cos\theta \\ +\sin\theta \\ 0 \end{pmatrix}$ $\quad x_2' = a_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ $\quad x_3' = a_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$A \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} \cos\theta & +\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

x_1 in new system $= A^T \hat{x}_1 = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$ ✓ x_1 expressed in the new, x' system

2) Show formally that the alternating tensor ε_{ijk} is a tensor; i.e. for any (orthogonal) transformation of the coordinate system given by a_{pq}, show that ε_{ijk} satisfies

$$\varepsilon_{ijk}' = a_{ip} a_{jq} a_{kr} \varepsilon_{pqr}$$

Definition of $\varepsilon_{ijk} = 1$ if have i,j,k as $(1,2,3)$; $(2,3,1)$; or $(3,1,2)$

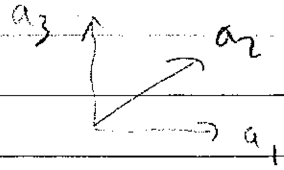
$= -1$ if anticyclic: $(3,2,1)$; $(2,1,3)$; $(1,3,2)$

$= 0$ if any two indices are identical

Consider an orthogonal set of coordinate axes a_1, a_2, a_3

Then by the definitions, $a_3 = a_1 \times a_2$; $a_1 = a_2 \times a_3$; $a_2 = a_3 \times a_1$

$-a_1 = a_3 \times a_2$ etc.



Consider $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ row a_1, row a_2, row a_3

Any row of an orthogonal set is the cross product ($\pm$) of the other rows. $a_i = a_j \times a_k$ if i, j, k are in cyclic order

Formally the cross product of a vector is defined as

$(v \times w)_i = \varepsilon_{ijk} v_j w_k \Rightarrow (v \times w)_p = \varepsilon_{pqr} v_q w_r$ let $\begin{pmatrix} v \to a_j \\ w \to a_k \end{pmatrix}$

So a_i is the cross product of rows a_j and a_k: $a_{ir} = (a_j \times a_k)_p = \varepsilon_{pqr} a_{jq} a_{kr}$

Let's take the dot product of row a_i with the cross product of $a_j \times a_k$:

Tutorial 2 problem 2 continued.

$\pm a_i$:

$\vec{a}_i \cdot (\vec{a}_j \times \vec{a}_k) \Rightarrow a_{ip}(a_j a_k)_p = a_{ip}\, a_{jq}\, a_{kr}\, \varepsilon_{pqr}$

But this is +1, −1, or 0 depending if:

if either j or k are same as i, = 0 since dot product of itself x another = 0

If i, j, k are cyclic, then $a_j \times a_k \Rightarrow +a_i = 1$

If j, k anticyclic then $a_j \times a_k \Rightarrow -a_i = -1$

$\therefore\ a_i \cdot (\vec{a}_j \times \vec{a}_k) = \varepsilon_{ijk} = a_{ip}\, a_{jq}\, a_{kr}\, \varepsilon_{pqr}$ QED

3) If a continuum is subject to a stress S_{ij} at a point P, find expression for the Normal + total Shear component of force across any plane through P.

From notes p. 27: Force $\vec{F}$ across a plane with normal $\hat{n}$ is:

$\vec{F} = S\hat{n}$ where $\vec{F}$ = force / unit area

$F_j = S_{jk} n_k$

Normal Stress force/unit area = $\vec{F} \cdot \hat{n} = F_j n_j = S_{jk} n_k n_j$

(for Force, multiply by area)

Shear stress = $(\text{Total}^2 - \text{Normal}^2)^{1/2}$

$|F|^2 = \vec{F} \cdot \vec{F} = F_j F_j = S_{jk} n_k\, S_{jl} n_l = (S_{jk} n_k)^2$

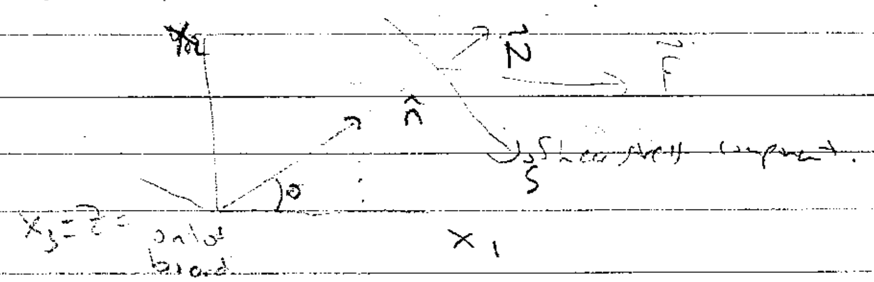
Shear stress = $\left((S_{jk} n_k)^2 - (S_{jk} n_k n_j)^2 \right)^{1/2}$

Or for the vector stress Traction: Use rejection of F on $\hat{n}$

Shear stress = $\vec{F} - (\vec{F} \cdot \hat{n})\hat{n}$

Tutorial 2 Problem 4.

4. $S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Find the normal N and total Shear Force S components across a plane w/ normal $\hat{n}^T = (\cos\theta, \sin\theta, 0)$



From previous problem Normal stress $N = \vec{F} \cdot \hat{n} = S_{jk} n_k n_j$

$$F = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{pmatrix} S_1 \cos\theta \\ S_2 \sin\theta \\ 0 \end{pmatrix}$$

$N = F \cdot \hat{n} = S_1 \cos^2\theta + S_2 \sin^2\theta$ in $\hat{n}$ direction

Shear stress $\vec{S}_T = \vec{F} - \vec{N} = \begin{pmatrix} S_1 \cos\theta \\ S_2 \sin\theta \\ 0 \end{pmatrix} - (S_1 \cos^2\theta + S_2 \sin^2\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

Let's ignore x_3 components since they are all 0.

Also, let's change $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

See Stein + expression

Show that (N, S) lie on a circle in the N, S plane (like x_1, x_2 axis) centred at $\left(\frac{S_1+S_2}{2}, 0\right)$ with radius $\frac{S_1-S_2}{2}$

(Just like first problem $\rightarrow$ / $a = S_1 + S_2$

- Such a circle would have eqn: $\left(x_1 - \left(\frac{S_1+S_2}{2}\right)\right)^2 + x_2^2 = \left(\frac{S_1-S_2}{2}\right)^2$

$\rightarrow$ Can rewrite $\vec{N} = S_1 \cos^2\theta + S_2 \sin^2\theta = S_1 \left(\frac{1+\cos 2\theta}{2}\right) + S_2 \left(\frac{1-\cos 2\theta}{2}\right)$

$\vec{N} = \left(\frac{S_1+S_2}{2}\right) + \left(\frac{S_1-S_2}{2}\right)\cos 2\theta$ $\rightarrow$ This is a circle with centre $\frac{S_1+S_2}{2}$ + radius $\frac{S_1-S_2}{2}$ $\Rightarrow$ θ goes from 0 to 180

$\vec{S}_T =$ $\rightarrow$ see next page

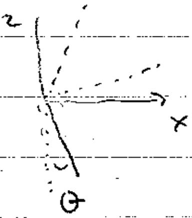
$$S_T = \vec{F} - \vec{N} = \begin{pmatrix} \left[S_1 - (S_1\cos^2\theta + S_2\sin^2\theta)\right]\cos\theta \\ \left[S_2 - (S_1\cos^2\theta + S_2\sin^2\theta)\right]\sin\theta \end{pmatrix} = \begin{pmatrix} \left(S_1\overbrace{(1-\cos^2\theta)}^{\sin^2\theta} - S_2\sin^2\theta\right)\cos\theta \\ \left(S_2\underbrace{(1-\sin^2\theta)}_{\cos^2\theta} - S_1\cos^2\theta\right)\sin\theta \end{pmatrix}$$

$$S_T = \begin{pmatrix} (S_1 - S_2)\sin^2\theta\cos\theta \\ (S_2 - S_1)\cos^2\theta\sin\theta \end{pmatrix} = (S_1 - S_2)\sin\theta\cos\theta \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

Use trig. relation $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\therefore$ $S_T = \dfrac{S_1 - S_2}{2}\sin 2\theta \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$ - a vector w/ magnitude $\frac{S_1 - S_2}{2}\sin 2\theta$

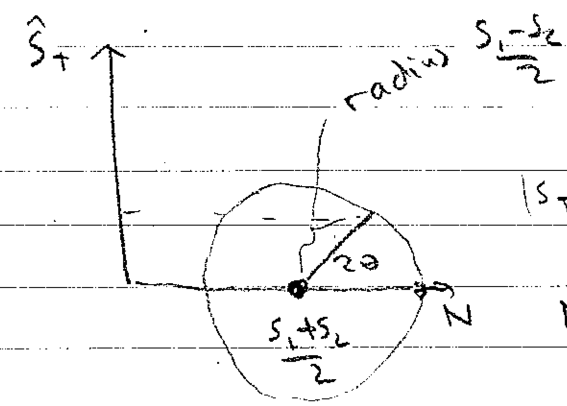
and direction θ measured from the $-x_2$ axis



$\hat{N}$ has θ measured from the x_1 axis

So $\hat{N}$, $\hat{S}_T$ are $\perp$ as expected, and their magnitudes are

$\left(\frac{S_1 - S_2}{2}\right)$ $\sin 2\theta$ or $\cos 2\theta$ So we can plot these in the τ / S plane as:



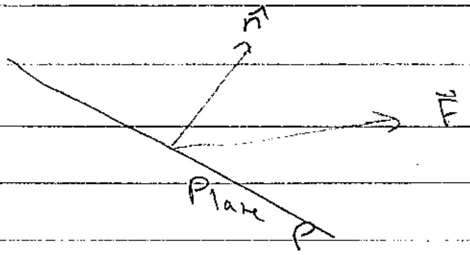
$$|S_T| = \left(\frac{S_1 - S_2}{2}\right)\sin 2\theta$$

$$N = \left(\frac{S_1 + S_2}{2}\right) + \left(\frac{S_1 - S_2}{2}\right)\cos 2\theta$$

This is the Mohr Circle. The maximum stress by inspection of the circle, are given when $\theta = 0$ for max Normal stress and when $2\theta = 90°$ for max tangential stress, or $\theta = 45°$.

Tutorial Two problem 5.

5) If $\vec{F}$ is the force exeted across a plane P, show that the stress force exerted across any plane that contains $\vec{F}$ lies in the plane of P.



$$\vec{F} = S\hat{n}$$

Any plane that contains $\vec{F}$ has normal

$\vec{N}\cdot\vec{F}=0$

So the stress force F' across any plane that contains $\vec{F}$ will be found by

$$F' = S\hat{N}$$

F' lines in the plane P if $F'\cdot\hat{n}=0$

so $F'\cdot\hat{n} = S\hat{N}\cdot\hat{n} = S\hat{n}\cdot\hat{N}$ by commutativity of dot product

$\therefore$ $F'\cdot\hat{n} = \vec{F}\cdot\hat{N} = 0$ QED