Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 1 Solutions	T1 2024

Due 3pm Monday 11 March

1. Prove Exercise 1.2: "Let G be a graph with a walk W from a vertex u to a vertex v. Then there is a path from u to v that uses a subset of the edges of W."

Solution. Say W is $u_1, e_1, u_2, e_2, \ldots, u_{n-1}, e_{n-1}, u_n$ where $u = u_1$ and $v = u_n$. If this is not a path, then there exists $i, j \in \{1, 2, \ldots, n\}$ with $i < j$ such that $u_i = u_j$. In this case

$$u_1, e_1, u_2, e_2, \ldots, u_i, e_j, u_{j+1}, e_{j+1}, u_{j+2}, \ldots, e_{n-1}, u_n$$

is also a walk from u to v, and this walk uses a proper subset of the edges of W. In particular, the length of this path is shorter than that of W. We can repeat this process for as long as we do not have a path. As each iteration reduces the length of the walk, we cannot keep iterating forever, so the process must eventually arrive at a path from u to v. □

The fact that a walk between vertices implies that there is a path between vertices is fundamental. We will use it many times in this course.

2. The *complement* of a simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$, where an edge xy (for any distinct $x, y \in V$) is in $\overline{E}$ if and only if xy is not in E. A simple graph is *self-complementary* if it is isomorphic to its complement. Prove:

(a) If G is a disconnected graph, then $\overline{G}$ is connected.

Solution. To see this, assume that G is disconnected. Then there is a partition $\{X, Y\}$ of $V(G)$ (with $X \neq \emptyset$ and $Y \neq \emptyset$) such that no edge of G joins a vertex in X to a vertex in Y; for example, take X to be the vertex set of one component C, and then let Y be the vertices of G that are not in C. This means that, in $\overline{G}$, there is an edge between every vertex in X and every vertex in Y.

We'll show that there is a path between any two vertices in $\overline{G}$. Say u and v are vertices of $\overline{G}$. If one is in X and the other in Y, then there is an edge between them, so there is certainly a walk from u to v. Otherwise we may assume, without loss of generality, both u and v are in X. As Y is non-empty, we can choose a vertex w in Y. Then u, w, v is the set of vertices in a path of length two from u to v.

We've now shown that there is a walk joining every pair of vertices in $\overline{G}$, so this graph is connected. This completes the proof. $\square$

(b) Every non-empty self-complementary graph is connected. (Hint: use (a).)

Solution. Suppose that G is a non-empty graph (so G has at least one vertex). If G is disconnected, then $\overline{G}$ is connected by (a), but then $G \neq \overline{G}$, so that G is not self-complementary. We have shown that if G is disconnected, then it is not self-complementary. Thus we have shown (via the contrapositive) that every non-empty self-complementary graph is connected. $\square$

(c) If G is self-complementary, then either $|V| \equiv 0 \bmod 4$ or $|V| \equiv 1 \bmod 4$.

Solution. Say G is self-complementary on n vertices. Then $|E| = |\overline{E}|$ and $|E| + |\overline{E}| = |E(K_n)| = n(n-1)/2$. Hence $|E| = n(n-1)/4$. For G to be self-complementary, $n(n-1)$ must be divisible by 4. Say $n = 4k + t$, where k is a non-negative integer and $t \in \{0, 1, 2, 3\}$. Then

$$n(n-1) = (4k+t)(4k+t-1) = 4(4k^2 + 2kt - k) + t(t-1)$$

which is divisible by 4 if and only if $t(t-1)$ is divisible by 4. By checking the value of $t(t-1)$ for each $t \in \{0, 1, 2, 3\}$, we see that $t \in \{0, 1\}$. That is, we have shown that G has either $4k$ or $4k+1$ vertices, for some non-negative integer k, as required. $\square$

3. Let s and t be positive integers with $s \leq t$. Recall that P_s is the path graph on s vertices. Give a formula for the minimum number of edges that need to be removed from K_t so that it has a graph isomorphic to P_s as an induced subgraph.

Solution. The path graph P_s has $s-1$ edges, whereas K_s has $s(s-1)/2$ edges. After choosing some set X of s vertices, on which we will look for the P_s induced subgraph, we must remove all but $s-1$ of the $s(s-1)/2$ edges in $G[X]$. That is, we need to remove at least

$$s(s-1)/2 - (s-1) = (s-1)(s-2)/2$$

edges. $\square$

4. The *distance* $d(u, v)$ between two vertices in a graph G is defined as the length of the shortest path that joins u and v. Prove that the distance satisfies the *triangle inequality*, that is, prove that $d(u, w) \leq d(u, v) + d(v, w)$ for any three vertices u, v and w of G.

Solution. Let $P(u, v)$ and $P(v, w)$ be shortest paths from u to v and v to w respectively. We obtain a walk $W(u, w)$ from u to w by combining $P(u, v)$ and $P(v, w)$ in an obvious way. Now the length of $W(u, w)$ is equal to $d(u, v) + d(v, w)$. Now $W(u, w)$ is walk, but it need not be a path. However, by Exercise 1.2 (see Q1) there is a subset of edges that induces a path. Therefore there *is* a path from u to w whose length is at most the length of $W(u, w)$. Hence $d(u, w) \leq d(u, v) + d(v, w)$ as required. $\square$

5. Let u be a vertex of odd degree in the graph G. Prove that there is a path from u to another vertex of odd degree in G.

Solution. Construct a random walk starting at u, with the constraint that we are never allowed to use an edge twice. Each time the walk enters a vertex of even degree it is possible to find an edge to leave the vertex and continue the walk. Eventually we must get stuck as G has only a finite number of edges. By the above observation we get stuck at a vertex, v say, of odd degree.

We have a walk from u to v, where v has odd degree. By Exercise 1.2 (see Q1), there is a path from u to v. $\square$

6. We know that trees with at least two vertices have at least two leaves. But typically trees have more leaves than that.

(a) Show that if a tree has a vertex of degree k, then it has at least k leaves.

Solution. Let G be a tree with a vertex v of degree k. Let $e_1, e_2, \ldots, e_k$ be the edges incident with v. For $i \in \{1, 2, \ldots, k\}$, let P_i be a maximal path that begins $v, e_i, \ldots$. Suppose that the path P_i ends at the vertex v_i. Then v_i has degree at least one, as it is adjacent to a vertex in P_i. Note that v_i is not adjacent to any other vertex in P_i, for otherwise G has a cycle, contradicting that G is a tree. If v_i has degree at least two, then we could extend the path P_i. But this contradicts that P_i is maximal. We deduce that v_i has degree one – that is, it is a leaf.

We need to show that the set $\{v_1, v_2, \ldots, v_k\}$ has k elements. If not, then there is an $i \neq j$ such that $v_i = v_j$. In this case the path P_i meets the path P_j at some vertex w. We now have a cycle in the tree by taking P_i from v to w and returning to v using the path P_j, contradicting the fact that trees have no cycles. Hence the elements of $\{v_1, v_2, \ldots, v_k\}$ are all distinct, so that the tree has at least k leaves. $\square$

(b) Let T be a tree with n vertices, k leaves, and a vertex with degree k, where $k \geq 2$. Suppose that $n > k + 1$. Prove that T has a vertex of degree two.

Solution. Let S be the set of vertices of T that are not leaves. Then $|S| = n - k > 1$. Let u be the vertex in S with degree k. Each other vertex in S

has degree at least two (since it is not a leaf). Thus we have

$$\sum_{v \in V(T)} d(v) \geq k + 2(|S| - 1) + k \tag{0.1}$$

$$= k + 2(n - k - 1) + k = 2n - 2.$$

Equality holds in (0.1) if and only if all vertices in $S \setminus \{u\}$ have degree 2. Since T is a tree with n vertices, it has $n - 1$ edges (by Theorem 2.5). So, by the Handshaking Lemma

$$\sum_{v \in V(T)} d(v) = 2(n - 1) = 2(n - 1).$$

This shows that equality does hold in (0.1), so each vertex in $S \setminus \{u\}$ has degree 2. In particular, since $|S| > 1$, there is at least one vertex in T with degree 2. □

7. A graph is *k-regular* if every vertex has degree k. Prove or disprove the following:

(a) If G is a k-regular bipartite graph, with $k \geq 2$, then G has no bridges.

Solution. This is true: we give a proof below.

Let G be a k-regular bipartite graph, with $k \geq 2$, and suppose that G has a bridge $e = uv$. Then $G \backslash e$ has a component C with $u \in V(C)$ and $v \notin V(C)$. The graph C is bipartite (since G is bipartite, it is 2-colourable, and this 2-colouring also induces a 2-colouring of C). Since C is bipartite, there exist disjoint sets A and B such that $A \cup B = V(C)$ and each edge of C has one end in A and the other end in B. Without loss of generality, assume $u \in A$. Consider the graph C. In this graph, the vertex u has degree $k - 1$, whereas every other vertex has degree k. Thus the sum of the degrees of the vertices in A is $k|A| - 1$, whereas the sum of the degrees of the vertices in B is $k|B|$. Since every edge of C joins a vertex in A to a vertex in B, we have $k|A| - 1 = k|B|$. So $k(|A| - |B|) = 1$. But $|A| - |B|$ and k are integers and $k \geq 2$, so this is a contradiction. This proves that if G is a k-regular bipartite graph with $k \geq 2$, then G has no bridges. □

(b) If G is a k-regular graph, with $k \geq 2$, then G has no bridges.

Solution. This is false. One counterexample can be obtained as follows. Start with K_4 and subdivide[1] a single edge. The resulting graph H has 4 vertices of degree 3, and a unique vertex with degree 2. Take two copies of this graph, and add an edge e between the two vertices with degree 2. Call the resulting graph G. Then e is a bridge of G, and G is 3-regular. □

[1] To subdivide an edge $e = uv$, we replace the edge e with a path of length two, via a new vertex, w say.