Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 1	T1 2024

Due 3pm Monday 11 March

1. Prove Exercise 1.2: "Let G be a graph with a walk W from a vertex u to a vertex v. Then there is a path from u to v that uses a subset of the edges of W."

2. The *complement* of a simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$, where an edge xy (for any distinct $x, y \in V$) is in $\overline{E}$ if and only if xy is not in E. A simple graph is *self-complementary* if it is isomorphic to its complement. Prove:

(a) If G is a disconnected graph, then $\overline{G}$ is connected.
(b) Every non-empty self-complementary graph is connected. (Hint: use (a).)
(c) If G is self-complementary, then either $|V| \equiv 0 \bmod 4$ or $|V| \equiv 1 \bmod 4$.

3. Let s and t be positive integers with $s \leq t$. Recall that P_s is the path graph on s vertices. Give a formula for the minimum number of edges that need to be removed from K_t so that it has a graph isomorphic to P_s as an induced subgraph.

4. The *distance* $d(u, v)$ between two vertices in a graph G is defined as the length of the shortest path that joins u and v. Prove that the distance satisfies the *triangle inequality*, that is, prove that $d(u, w) \leq d(u, v) + d(v, w)$ for any three vertices u, v and w of G.

5. Let u be a vertex of odd degree in the graph G. Prove that there is a path from u to another vertex of odd degree in G.

6. We know that trees with at least two vertices have at least two leaves. But typically trees have more leaves than that.

(a) Show that if a tree has a vertex of degree k, then it has at least k leaves.
(b) Let T be a tree with n vertices, k leaves, and a vertex with degree k, where $k \geq 2$. Suppose that $n > k + 1$. Prove that T has a vertex of degree two.

7. A graph is *k-regular* if every vertex has degree k. Prove or disprove the following:

(a) If G is a k-regular bipartite graph, with $k \geq 2$, then G has no bridges.
(b) If G is a k-regular graph, with $k \geq 2$, then G has no bridges.

Tutorial exercises:

1. Prove Theorem 1.5: "For any graph G with vertex set V, the relation $\sim$ is an equivalence relation on V."

2. Prove Theorem 1.6: "For each positive integer n, the complete graph K_n has $n(n-1)/2$ edges."

3. The *complement* of a simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$, where an edge xy (for any distinct $x, y \in V$) is in $\overline{E}$ if and only if xy is not in E. A simple graph is *self-complementary* if it is isomorphic to its complement.

(a) Give an example of a graph that is self-complementary.
(b) Prove that every self-complementary graph with $4k + 1$ vertices has a vertex of degree $2k$ (where k is a non-negative integer).

4. Let s and t be positive integers. Recall that P_s is the path graph on s vertices.

(a) For what values of s and t (if any) is P_s an induced subgraph of K_t?
(b) For what values of s and t is P_s a subgraph of K_t?
(c) For the values of s and t given in (b), provide a formula for the number of edges that need to be removed from K_t to obtain P_s as a subgraph.

5. Recall that an *Eulerian walk* in a graph is a walk that uses each edge exactly once. Recall that a connected graph has an Eulerian walk if and only if it has either 0 or 2 vertices of odd degree. What happens if, instead, we look for a walk that uses each edge exactly twice — once in each direction? Prove or disprove that every connected graph contains such a walk.

6. Prove Lemma 2.3: "Let T be a tree with at least two vertices. Then T has at least two leaves."

7. Prove Lemma 2.4: "Let G be a connected graph. If e is a bridge of G, then $G \backslash e$ has exactly two components."

8. A *saturated hydrocarbon* is a molecule $C_m H_n$ is which every carbon atom has four bonds, every hydrogen atom has one bond, and no sequence of bonds forms a cycle. Show that, for any positive integer m, the molecule $C_m H_n$ can exist if and only if $n = 2m + 2$.

9. It is a famous theorem in graph theory that K_n has n^{n-2} spanning trees.

(a) Verify the theorem for K_2, K_3 and K_4.
(b) What about K_1?
(c) Can you think of strategies for proving the theorem in general?