Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 2 Solutions	T1 2024

Due 3pm Tuesday 26 March

1. (Exercise 2.7) Prove Theorem 2.6: "Let G be a forest with n vertices and c components. Then G has $n - c$ edges."

Solution. We use induction on the number of edges in G. First, observe that the theorem holds when G has no edges since, in this case, each vertex is in a component by itself, in which case the number of vertices, n, equals the number of components, c, so $n - c = 0$.

Now suppose G has m edges, where $m \geq 1$, and that the theorem holds for any forest with $m - 1$ edges. Let H be a component of G with at least one edge. Then H is a tree with at least two vertices, so H has a leaf (by Lemma 2.3). Let v be a leaf of H, and let e be the pendant edge incident with v. Now, $G - v$ has $n - 1$ vertices, and $m - 1$ edges since v has degree one. By Lemma 2.14, v is not a cut vertex, so $G - v$ has c components (as G has c components). By the induction assumption, $G - v$ has $(n - 1) - c$ edges. As G has one extra edge, e, it has $n - c$ edges, as required. The result follows by induction. □

Note: the proof of Theorem 2.5 in lectures (and the course notes) used strong induction on n, whereas here we use (ordinary) induction on m. In particular, in order to apply the induction assumption, we pick a leaf to delete, so that the number of edges is decreased by exactly one.

2. Prove Lemma 3.1:

> "Let G be a connected graph, and let e be an edge of G. Then G/e is connected."

(Hint: You might like to use Lemma 2.10, which appears in Tutorial Q4.)

Solution. Say e is a loop of G. As G is connected, for any pair of vertices x and y in G, there is a path from x to y. As e is a loop, this path does not contain e and it remains a path in G/e (recalling that $G/e \cong G \backslash e$). Hence G/e is connected.

Now assume that e is not a loop of G. By Lemma 2.10, there is a spanning tree of G that contains e. Let T be the edges of this spanning tree. Then, by Lemma 2.13, $T \setminus \{e\}$ is the set of edges of a spanning tree of G/e. As G/e has a spanning tree, this graph is connected (by Theorem 2.8). □

3. (Exercise 3.6) Prove Lemma 3.5: "Let G be a 2-connected graph. If e and f are parallel edges in G, then $G\backslash e$ is 2-connected."

Note: it may be tempting to use Theorem 3.7 here, but this is a a cheat since we used Lemma 3.5 to prove Theorem 3.7.

Solution. Suppose that e and f are parallel edges in G. To show that $G\backslash e$ is 2-connected, we will show that $G\backslash e$ has at least three vertices, it is connected, and it has no cut vertices.

Firstly, since G is 2-connected, it has at least three vertices, so $G\backslash e$ does too.

Secondly, since G is connected, there is a path between every pair of vertices in G. If this path contains e, then we can replace it with f in order to obtain a path in $G\backslash e$. This shows that $G\backslash e$ is connected.

Finally, towards a contradiction, suppose that $G\backslash e$ has a cut vertex v. Then $G\backslash e$ is connected but $(G\backslash e) - v$ is disconnected. Let a and b be vertices in different components of $(G\backslash e) - v$. Then there is no path from a to b in $(G\backslash e) - v$.

Since G is 2-connected, $G - v$ is connected (see Exercise 3.3). So there is a path from a to b in $G - v$. If e is in this path, we can replace it with f, and thereby obtain a path from a to b in $(G\backslash e) - v$. But we saw (in the previous paragraph) that no such path exists, so this is a contradiction. Therefore, $G\backslash e$ has no cut vertices. This completes the proof that $G\backslash e$ is 2-connected. □

4. Prove Corollary 3.8: "Let u and v be vertices of a 2-connected graph G. Then there is a cycle of G that contains both u and v." (Hint: you may use Theorem 3.7.)

Solution. We may assume that G is loopless (as if it holds for the graph obtained from G by deleting all the loops, then it also holds for G). Since G is 2-connected, it has at least three vertices, and is connected. Therefore, by Theorem 2.8, it has a spanning tree, and so G has at least two edges (by Theorem 2.5). If G has a leaf, then it has a pendant edge e, and there is no cycle containing e and any other edge of G, which contradicts Theorem 3.7. This shows that every vertex of G has degree at least two. Let e and f be distinct edges incident with u and v respectively (such edges exist, even in the case that $u = v$, since each vertex has degree at least two). By Theorem 3.7, there is a cycle C of G that contains both e and f. Then C contains both u and v. □

5. (Exercise 3.10) Prove that if G_1 and G_2 are consistent connected graphs with at least one vertex in common, then $G_1 \cup G_2$ is connected.

Solution. We need to show that there is a walk between every pair of vertices in $G_1 \cup G_2$. Let u and v be any two vertices in $G_1 \cup G_2$. If u and v are both in G_1, then there is a path (and walk) between u and v in G_1, since G_1 is connected, and this is also a path (and walk) in $G_1 \cup G_2$. Similarly, there is a walk between u and v in $G_1 \cup G_2$ when $\{u, v\} \subseteq V(G_2)$. So suppose, without loss of generality, $u \in V(G_1)$ and $v \in V(G_2)$. Let w be a vertex in $V(G_1) \cap V(G_2)$. Then there is a path from u to w in G_1 (since u and w are vertices of G_1 and G_1 is connected), and similarly a path from w to v in G_2. Concatenating these paths, we have a walk from u to v in $G_1 \cup G_2$. Since there is a walk between every pair of vertices in $G_1 \cup G_2$, this graph is connected, as required. □

6. (Exercise 3.15(i)) Prove Lemma 3.14(i):

> Let G be a loopless graph, and let $B(G)$ be the block-cut graph of G. Then $B(G)$ is a forest.

Solution. Towards a contradiction, suppose $B(G)$ contains a cycle. Since $B(G)$ is bipartite and simple, any cycle has length at least four. Thus, for some $t \geq 2$, there is a sequence $v_1, B_1, v_2, B_2, \dots, v_t, B_t, v_1$ such that B_i contains $\{v_i, v_{i+1}\}$ for $i \in \{1, 2, \dots, t-1\}$, and B_t contains $\{v_1, v_t\}$. Since each block B_i is connected, by definition, there is a path between any pair of vertices in B_i. Thus, there is a path P_i in B_i from v_i to v_{i+1} for each $i \in \{1, 2, \dots, t-1\}$, and also a path P_t from v_t to v_1 in B_t. The concatenation of P_1, P_2, …, P_t is now a closed walk in G (recall, a *closed* walk is a walk that begins and ends at the same vertex). Let C be a cycle that is contained in this closed walk. Let e be an arbitrary edge in C. Then e is an edge of the path P_i for some $i \in \{1, 2, \dots, t\}$. Moreover, since P_i is a path but C is a cycle, we can find an edge f of C that is not in the path P_i. Recalling that each edge of G belongs to a unique block (by Lemma 3.9(ii)), we have that e is in the block B_i, but f is not in the block B_i, where C contains both e and f. The cycle C is a biconnected subgraph of G, so it is contained in a block B of G. Since e belongs to B and B_i, we have $B = B_i$ (by Lemma 3.9(ii)), but this contradicts that f is an edge of B that is not in B_i. From this contradiction, we deduce that $B(G)$ has no cycles, so $B(G)$ is a forest. □

7. Let G be a loopless graph. We say that a block of G is a *leaf block* if it contains precisely one cut vertex of G. Prove that every loopless connected graph with at least three vertices that is not 2-connected has at least two distinct leaf blocks.

Solution. Let G be a graph that is connected but not 2-connected, and $|V(G)| \geq 3$. Consider the block-cut graph $B(G)$ of G. By Lemma 3.14(i) (see Q6), $B(G)$ is a forest. Since G is not 2-connected and $|V(G)| \geq 3$, it has at least one cut vertex v. Consider the component T of $B(G)$ containing v (here, we could use Lemma 3.14(ii) that says that $B(G)$ has just one component since G is connected, but I will bypass this since it is not something we have proved). Each cut vertex of G belongs to at least two blocks (see Lemma 3.12 or Tutorial Q9), so each vertex of $B(G)$ that corresponds to a cut vertex has degree at least two. In particular, v has degree at least two in T, so T is a tree on at least three vertices. Therefore T has at least two leaves (by Lemma 2.3). As each cut vertex of G corresponds to a vertex of degree at least two in $B(G)$, these two leaves correspond to blocks of G that contain precisely one cut vertex. Thus G has at least two leaf blocks. $\square$