Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 2	T1 2024

Due 3pm Tuesday 26 March

1. (Exercise 2.7) Prove Theorem 2.6: "Let G be a forest with n vertices and c components. Then G has $n - c$ edges."

2. Prove Lemma 3.1:

> "Let G be a connected graph, and let e be an edge of G. Then G/e is connected."

(Hint: You might like to use Lemma 2.10, which appears in Tutorial Q4.)

3. (Exercise 3.6) Prove Lemma 3.5: "Let G be a 2-connected graph. If e and f are parallel edges in G, then $G\backslash e$ is 2-connected."

4. Prove Corollary 3.8: "Let u and v be vertices of a 2-connected graph G. Then there is a cycle of G that contains both u and v." (Hint: you may use Theorem 3.7.)

5. (Exercise 3.10) Prove that if G_1 and G_2 are consistent connected graphs with at least one vertex in common, then $G_1 \cup G_2$ is connected.

6. (Exercise 3.15(i)) Prove Lemma 3.14(i):

> Let G be a loopless graph, and let $B(G)$ be the block-cut graph of G. Then $B(G)$ is a forest.

7. Let G be a loopless graph. We say that a block of G is a *leaf block* if it contains precisely one cut vertex of G. Prove that every loopless connected graph with at least three vertices that is not 2-connected has at least two distinct leaf blocks.

Tutorial exercises:

1. Exercise 2.17 was "Describe exactly when a vertex incident with a bridge is not a cut vertex." I claimed to give an answer to this in lectures, but without a formal proof. Prove this statement:

> "Let u be a vertex in a graph G that is incident with a bridge, but is not a cut vertex. Then either u is a leaf of G, or u becomes a leaf when all loops incident to u are deleted."

2. A path Q in a graph G is *induced* if the vertices of Q induce a path graph. (That is, if Q is a path, and we denote the vertices of Q as $V(Q)$, then Q is *induced* if $G[V(Q)]$ is isomorphic to P_t for some positive integer t.) Let G be a simple connected graph with vertices u and w. Let P be a shortest path between u and w. Prove that P is an induced path.

3. Prove the following:

> "Let G be a simple graph. The graph G has no induced subgraph isomorphic to P_3 if and only if every component of G is isomorphic to a complete graph."

(Hint: It may be helpful to use the previous question.)

4. Prove Lemma 2.10: "Let G be a connected graph, and let e be an edge of G that is not a loop. Then there is a spanning tree of G that contains e."

5. (Exercise 3.2) Give an example of a 2-connected graph G with at least four vertices, and a vertex v such that $G - v$ is not 2-connected.

6. (Exercise 3.3) Prove that for a 2-connected graph G, the graph $G - v$ is connected for any $v \in V(G)$.

7. Prove the direction of Theorem 3.7 not given in the course notes. It is sufficient to prove the following statement:

> "Let G be a loopless graph with at least three vertices and no isolated vertices. If G is not 2-connected, then there exists a pair $\{e, f\}$ of edges of G such that no cycle contains both e and f."

8. (Exercise 3.11(i)) Prove Lemma 3.9(i):

> Let G be a loopless graph. If B_1 and B_2 are distinct blocks in G, then B_1 and B_2 have at most one vertex in common.

(Hint: Assignment Q5 might be useful.)

9. (Exercise 3.13) Prove Lemma 3.12:

> Let G be a loopless connected graph with at least one cut vertex. Then every cut vertex of G belongs to at least two distinct blocks.