Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 4 Solutions	T1 2024

Due 3pm Tuesday 14 May

1. Recall that Euler's formula (as we saw in lectures as Theorem 5.5) concerns *connected* plane graphs.

(a) Give an example to demonstrate that Euler's formula does not hold if G is not connected.

Solution. Let G be the graph on two vertices with no edges. This graph has a single face, the outer face, so $v(G) = 2$, $e(G) = 0$, $f(G) = 1$, and $v(G) - e(G) + f = 3$. □

(b) Let G be a plane graph, and let $c(G)$ be the number of components of G. State a generalisation of Euler's Formula that describes the relationship between $v(G)$, $e(G)$, $f(G)$ and $c(G)$. Prove this generalisation.

Solution. From the previous example, we might guess that for each extra component we have, we increase $v(G) - e(G) + f(G)$ by one. When we have $c(G) = 1$, we know from Euler's formula that $v(G) - e(G) + f(G) = 2$. So, let's conjecture that

$$v(G) - e(G) + f(G) = 1 + c(G).$$

To prove this, let's mimic the proof we saw in lectures; that is, the proof is by induction on the number of edges.

If G has no edges, then G consists of a $v(G)$ isolated vertices, so G has $v(G)$ components and a single face (the outer face). In this case, we have

$$v(G) - e(G) + f(G) = c(G) - 0 + 1,$$

as required. So the base case holds.

Assume now that G has at least one edge, and that the result holds for graphs with fewer than $e(G)$ edges. Let e be an edge of G. First suppose that e is not a loop. Recall that we saw (as Lemma 4.11) that the faces of G/e either correspond to faces of G whose boundary does not contain e, in which case their face boundary is the same in G/e; or, they correspond to a face of G whose boundary does contain e, in which case their face boundary in G/e is like in G but with e removed. In any case, we have

$$f(G/e) = f(G).$$

Also, since e is not a loop,

$$v(G/e) = v(G) - 1$$

and

$$e(G/e) = e(G) - 1.$$

Finally, note that we cannot increase the number of components by contracting an edge (as a consequence of Lemma 3.1), so we have $c(G/e) = c(G)$.

By the induction assumption,

$$v(G/e) - e(G/e) + f(G/e) = c(G/e) + 1.$$

Hence we have

$$(v(G) - 1) - (e(G) - 1) + f(G) = c(G) + 1,$$

that is,

$$v(G) - e(G) + f(G) = c(G) + 1,$$

as required.

On the other hand, when e is a loop, then $v(G/e) = v(G)$, $f(G/e) = f(G) - 1$, and $e(G/e) = e(G) - 1$. We still have $c(G/e) = c(G)$. Hence, by a similar argument,

$$v(G/e) - e(G/e) + f(G/e) = 1 + c(G/e),$$

so

$$v(G) - (e(G) - 1) + (f(G) - 1) = 1 + c(G),$$

implying

$$v(G) - e(G) + f(G) = 1 + c(G).$$

The result now follows by induction. □

2. Consider the following graph classes:

(a) The class of graphs with at most one cycle.
(b) The class of *pseudotrees*, consisting of connected graphs with at most one cycle.
(c) The class of *pseudoforests*, consisting of graphs where each component has at most one cycle.

For each of graph classes (a)–(c), answer the following questions:

(i) Is the class minor-closed?
(ii) If yes, what are the excluded minors? If no, give an explicit example to demonstrate this.

Solution. (a) (i) This class is closed under minors (since deleting an edge or contracting an edge cannot increase the number of cycles). (ii) Let G_1 be a graph with one vertex and two loops incident to that vertex, and let G_2 be the (disconnected) graph with two vertices and a single loop incident to each vertex. Then G_1 and G_2 are excluded minors. For any graph with at least two cycles, we can contract all but one edge of each cycle, and then delete all other edges, thereby obtaining either G_1 or G_2. That is, any graph that is not in the class has G_1 or G_2 as a minor. So these are the only two excluded minors.

(b) (i) This class is not closed under minors. (ii) P_3 is in the class, but by deleting the (unique) non-pendant edge we obtain a graph that is not connected, so is not in the class.

(c) (i) This class is closed under minors. (ii) The graph G_1 from (a) is the only excluded minor. To see this, observe that if a graph is not in the class, then it has a component with at least two cycles, and this component has G_1 as a minor.

□

3. Say that for any pair of people, they either both know each other, or they are strangers (neither knows each other). Prove that in any party of six people, there is a group of three that either all know each other, or all are strangers.

(Hint: how can you model this problem using a graph?)

Solution. A useful way to model this problem is to think of the people as being vertices of K_6. An edge joining two people is coloured green if they know each other and red if they do not. The problem then is to show that, in a complete graph on six vertices whose edges are coloured red or green, there is either a green triangle (three people who know each other) or a red triangle (three mutual strangers).

Choose a vertex a. This vertex has degree 5. It follows that there are either (at least) three green edges incident with a, or at least three red edges incident with a. Without loss of generality, assume the former holds. Then we have vertices b, c, d, say, such that ab, ac, and ad are all green. If bc is green, then abc is a green triangle. Similarly, if either cd or bd is green, then we have a green triangle. Therefore, the only way we can avoid having a green triangle is if bc, bd, and cd are all red. But then we have a red triangle on $\{b, c, d\}$. We have now shown that a complete graph on six vertices whose edges are coloured red or green contains either a red triangle or a green triangle, as required. □

4. Before attempting this question, see Tutorial Q6. As in that question, let $\mathcal{H}$ be the smallest class that contains the graphs K_5' and $K_{3,3}'$, and is closed under isomorphism and 2-sum. Prove that every graph in $\mathcal{H}$ is planar.

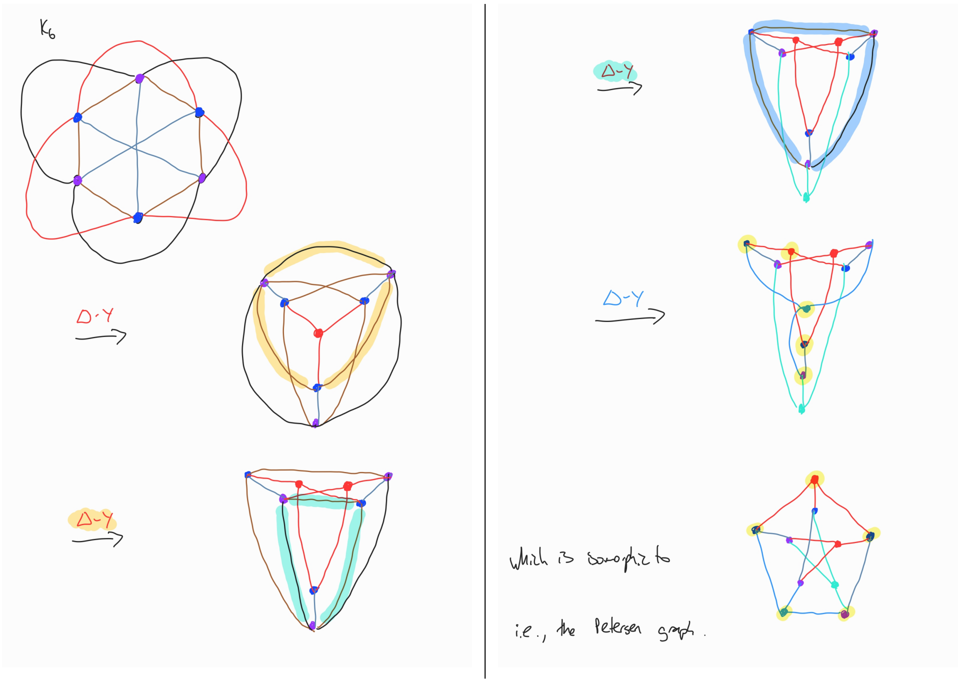
Solution. Let H be a graph in $\mathcal{H}$. We want to show that H is planar. We prove this by strong induction on the number of vertices of the graph H. Note that the only two graphs in $\mathcal{H}$ with at most 6 vertices are isomorphic to either K_5' or $K_{3,3}'$ (since the 2-sum of two graphs with at least five vertices has at least 8 vertices). We know that K_5' and $K_{3,3}'$ are planar (it is easy to find planar embeddings; or, we can just observe this as a consequence of Wagner's theorem, as they do not a K_5- or $K_{3,3}$-minor).

Now, assume that H is a graph in $\mathcal{H}$ with more than six vertices, and any graph in $\mathcal{H}$ with fewer than $|V(H)|$ vertices is planar. Since H has more than six vertices, it is obtained by the 2-sum of two graphs in $\mathcal{H}$. Say $H = H_1 \oplus_2 H_2$, where e is the single edge that H_1 and H_2 have in common. By the induction assumption, H_1 and H_2 are planar.

Let u and v be the ends of e. From the definition of 2-sum, we have $V(H) = V(H_1) \cup V(H_2)$, and there are no edges in H having one end in $V(H_1) \setminus \{u, v\}$ and the other end in $V(H_2) \setminus \{u, v\}$. This shows that $\{V(H_1), V(H_2)\}$ is a separation of H. Since $V(H_1) \cap V(H_2) = \{u, v\}$, this separation has order 2. By Tutorial Q3, H is 2-connected. Letting $A = V(H_1)$ and $B = V(H_2)$, so that $\{A, B\}$ is a proper separation of H of order two, $H_1 = G_A$ and $H_2 = G_B$ as defined just prior to Lemma 5.15 (see lecture 19). Thus, by Lemma 5.16, as H_1 and H_2 are planar, H is also planar, as required. $\square$

5. Show that K_6 is ΔY equivalent to the Petersen graph.

Solution. K_6 has six vertices, each with degree five, whereas the Petersen graph has 10 vertices, each with degree three. Each ΔY exchange introduces a new vertex with degree three, and reduces the degree of three existing vertices by one. So, starting from K_6, we'll be looking to perform four ΔY operations to obtain the Petersen graph. Indeed we can do this as shown below:



$\square$