Victoria University of Wellington
School of Mathematics and Statistics

| MATH 361 | Assignment 5 | T1 2024 |
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Due 3pm Tuesday 28 May

**1.** Prove Lemma 6.1: "A graph is $k$-colourable if and only if it is $k$-partite."

**2.** (Exercise 6.2) A university wishes to timetable its examinations. To prevent the examination period from being too long it attempts to do this with as few time slots as possible. The exams for two courses cannot be scheduled at the same time if there is a student enrolled in both courses. Explain why this problem is equivalent to the problem of finding the chromatic number of a certain graph.

**3.** Let $G$ be a simple connected graph. (Hint: Tutorial Q3 may be helpful for the following questions.)

(a) Suppose that $G$ has $t$ blocks. Prove that there is an ordering $(B_1, B_2, \ldots, B_t)$ of the blocks of $G$ such that, for each $i \in \{2, 3, \ldots, t\}$, the graphs $B_i$ and $B_1 \cup B_2 \cup \cdots \cup B_{i-1}$ have exactly one vertex in common.

(b) Prove that the chromatic number of $G$ is the maximum of the chromatic number of each block of $G$.

**4.** A graph is $k$-*degenerate* if it can be reduced to the empty graph by repeatedly deleting vertices of degree at most $k$.

(a) Prove that a graph $G$ is $k$-degenerate if and only if every subgraph of $G$ has a vertex of degree at most $k$.

(b) Prove that a graph is 1-degenerate if and only if it is a forest.

(c) Prove that every loopless $k$-degenerate graph is $(k+1)$-colourable.

**5.** Prove Lemma 6.26: "Every 3-connected graph has a $K_4$-minor."

## Tutorial exercises:

**1.** (Exercise 6.3) A company manufactures a number of chemicals, say $C_1, C_2, \dots, C_n$. Some chemicals cause explosions if they come into contact with each other. The company has a warehouse to store chemicals which it is planning to divide into rooms. Explosions, while indeed scientifically interesting, are perhaps best avoided. Hence we want to avoid storing incompatible chemicals in the same room. Explain why the problem of finding the minimum number of rooms is equivalent to the problem of finding the chromatic number of a graph.

**2.** Let $G$ be a simple graph. Prove that the chromatic number of $G$ is the maximum of the chromatic number of each component of $G$.

**3.** (Exercise 3.15(ii)) Let $G$ be a loopless graph. Recall that $B(G)$ denotes the block-cut graph of $G$, and (as proved in Assignment 2 Q6) $B(G)$ is a forest. Prove that if $G$ is connected, then $B(G)$ is a tree.

**4.** (Exercise 6.4) Find an efficient algorithm to decide if a graph is 2-colourable.

**5.** (Exercise 6.5) How many cycles does $K_n$ have?

**6.** Prove Lemma 6.14: "Every simple non-empty planar graph has a vertex of degree at most five."

**7.** Prove Lemma 6.18: "Every loopless planar graph is 4-colourable if and only if every triangulation is 4-colourable."

**8.** Let $G_1$ and $G_2$ be consistent graphs, let $G = G_1 \cup G_2$, and let $k$ and $\ell$ be positive integers.

(a) Prove that if $G_1$ is $k$-colourable and $G_2$ is $\ell$-colourable, then $G$ is $k\ell$-colourable.
(b) Give an example where, for integers $k$ and $\ell$ at least two, the graph $G_1$ is $k$-colourable, the graph $G_2$ is $\ell$-colourable, and the chromatic number of $G$ is $k\ell$.
(c) Extend the example from (b) to an infinite family of examples, where $k\ell$ can be arbitrarily large.

**9.** Let $G$ be a simple graph. Using Ramsey's theorem, prove that for any positive integer $s$, there exists a number $n$ such that if $G$ has a vertex of degree at least $n$, then $G$ has either $K_{s+1}$ or $K_{1,s}$ as an induced subgraph.