MATH361 | Lecture 13

* Assignment 2 due 3pm tomorrow
* Test on Mon 15 April

Planar Graphs I

A planar embedding of a graph G is an embedding in the plane (i.e. functions mapping $V(G)$ and $E(G)$ to the plane) where

$G = (V, E)$

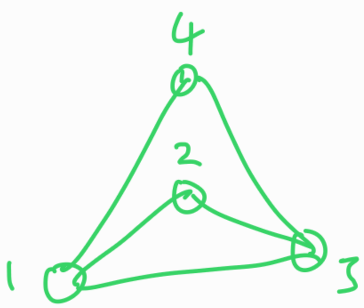
* vertices of G are mapped to points,
* edges of G are mapped to simple curves between the points of the incident vertices,
* the only points where edges meet is at vertices of G.

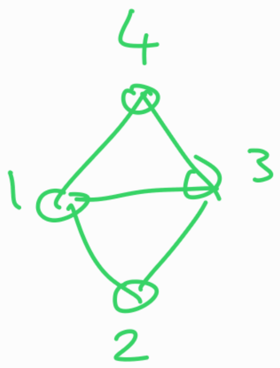
A graph is planar if it has a planar embedding.

A plane graph is a graph H together with a planar embedding of H.

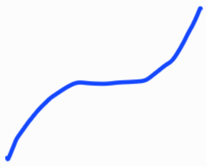
A plane graph is a graph with extra structure — the underlying graph is the graph we get when we ignore the extra structure.

eg.





Two plane graphs with the same underlying graph.

curve 

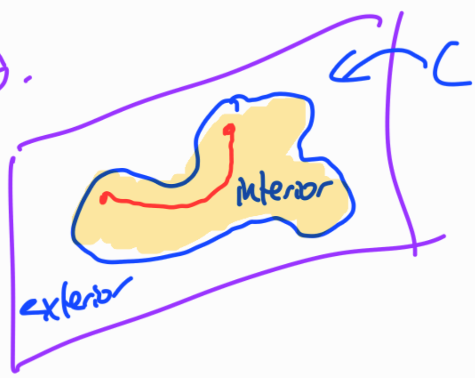
closed curve 

a (closed) curve is simple if it doesn't intersect itself.

Lemma 4.1: The edges of a cycle in a plane graph form a simple closed curve.

A subset X of the plane is arcwise-connected if any two points in X can be joined by a curve contained entirely in X.

e.g.



Theorem 4.2 (The Jordan Curve theorem).

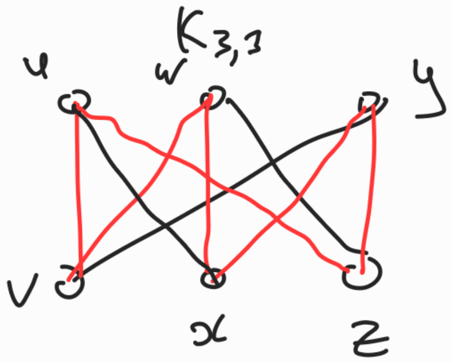
Any simple closed curve in the plane partitions the plane into two disjoint arcwise-connected open sets.

We call these two regions the interior and exterior.

To prove a graph is planar, it suffices to give a planar embedding.

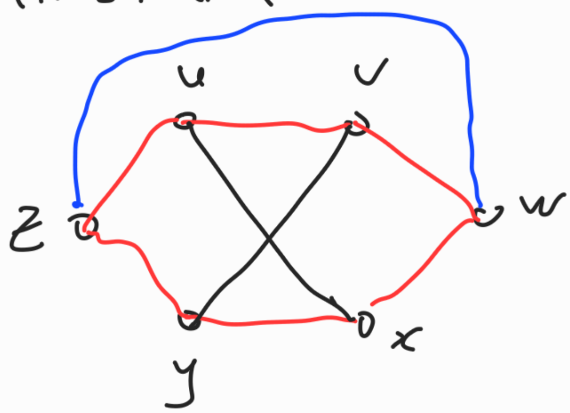
Theorem 4.3:

$K_{3,3}$ is not planar.



Proof: Consider the following labelling ↗ of $K_{3,3}$.

There is a Hamiltonian cycle as illustrated in red (along u, v, w, x, y, z, u).



By Lemma 4.1, the edges of this cycle will be a simple closed curve in any planar embedding of $K_{3,3}$. Suppose there is a planar embedding of $K_{3,3}$, and let C be this simple closed curve. By the Jordan Curve Theorem, C partitions the rest of the plane into 2 arcwise-connected regions, the interior and the exterior. The edges ux, vy, wz must lie in either the interior or the exterior. So (at least) two edges must lie in the same region and these must cross. From this contradiction we deduce that $K_{3,3}$ is not planar. □

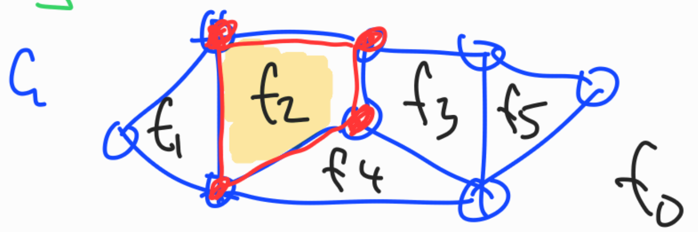
Exercise 4.4: Show K_5 is not planar.

Faces As a consequence of Lemma 4.1 and the Jordan Curve theorem, for a plane graph G, the edges of G partition the rest of the plane into arcwise-connected regions – we call these faces of G, denoted $F(G)$.

One of these faces is unbounded: we call this the outer face

The boundary of a face is the boundary (in the topological sense) of the open set – it corresponds to a closed walk

e.g.



$F(G) = \{f_0, f_1, \ldots, f_5\}$.

f_0 is the outer face

e.g. the boundary of f_2 is in red.