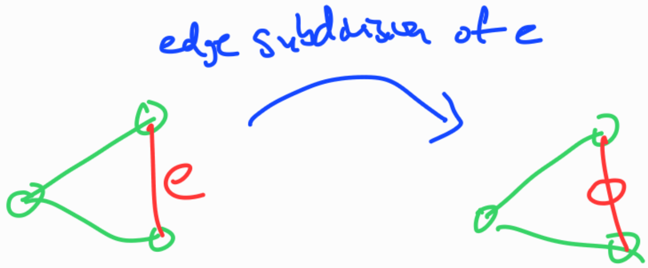
MATH361 | Lecture 22

Last time: • subdivision



e.g.

H is a subdivision of G.

Kuratowski's theorem:

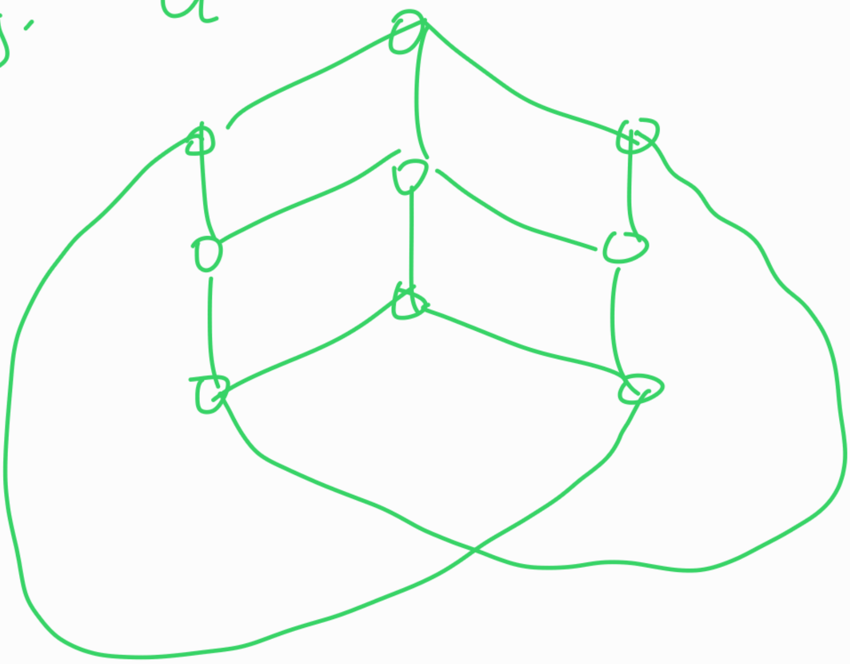
A graph is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$.

A graph H is a topological minor of a graph G if G contains a subgraph that is a subdivision of H.

Kuratowski's theorem, restated:

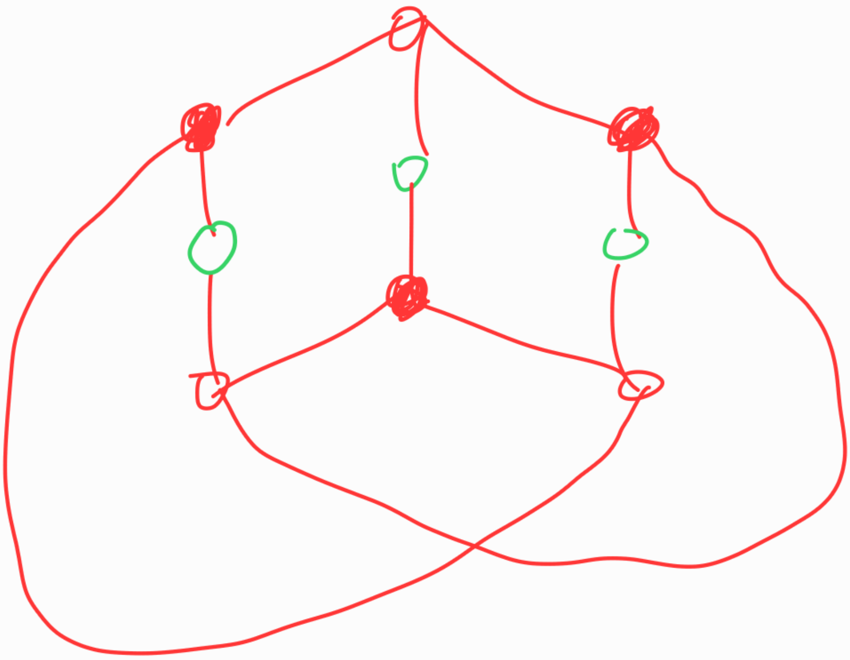
A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a topological minor.

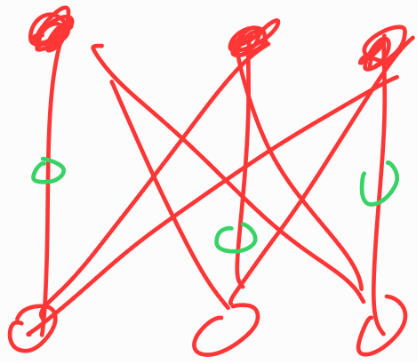
e.g. G



Does G have $K_{3,3}$ as a topological minor?

Yes

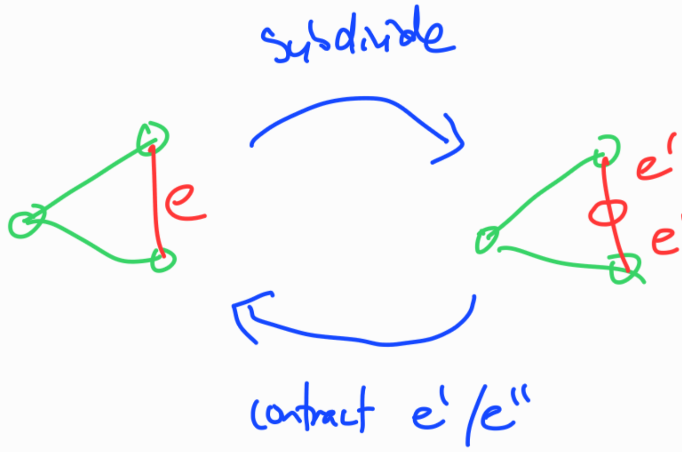




Hence G is not planar (by Kuratowski's Theorem).

<u>Lemma 5.17</u>: If H is a topological minor of a graph G, then H is a minor of G.

Proof: Say H is a topological minor of G.
Then H has a subdivision G' that is a subgraph of G.
So G' is a minor of G.
Moreover, H is a minor of G', as it can be obtained by contracting edges incident to degree-2 vertices (that came from a subdivided edge). Hence H is a minor of G. $\square$



Exercise 5.18: show that the converse doesn't always hold.

The converse of Lemma 5.17 does hold for a particular class of graphs, however.

A graph is cubic if every vertex has degree 3.

A graph is subcubic ———— degree at most 3.

Theorem 5.19: Let H be a cubic graph and let G be a graph that has H as a minor. Then G has H as a topological minor.

Vertex splitting

We say $\{A, B\}$ is a cover of X if $A \cup B = X$.

e.g. a separation $\{Y, Z\}$ is a cover of $V(G)$.

a bipartition $\{Y, Z\}$ of X is also a cover of X.

Let G be a graph with a vertex w of degree at least 2.

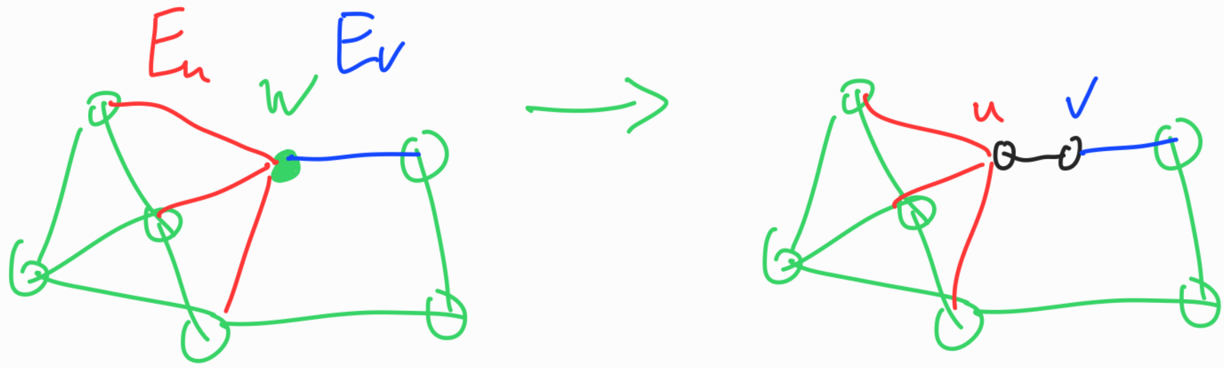
Let $\{E_u, E_v\}$ be a cover of the edges incident to w, such that each non-loop appears in one of E_u and E_v, and E_u and E_v are non-empty.

A vertex split at w corresponding to $\{E_u, E_v\}$ is obtained by removing w, introducing new vertices u and v where u is incident to the edges in E_u

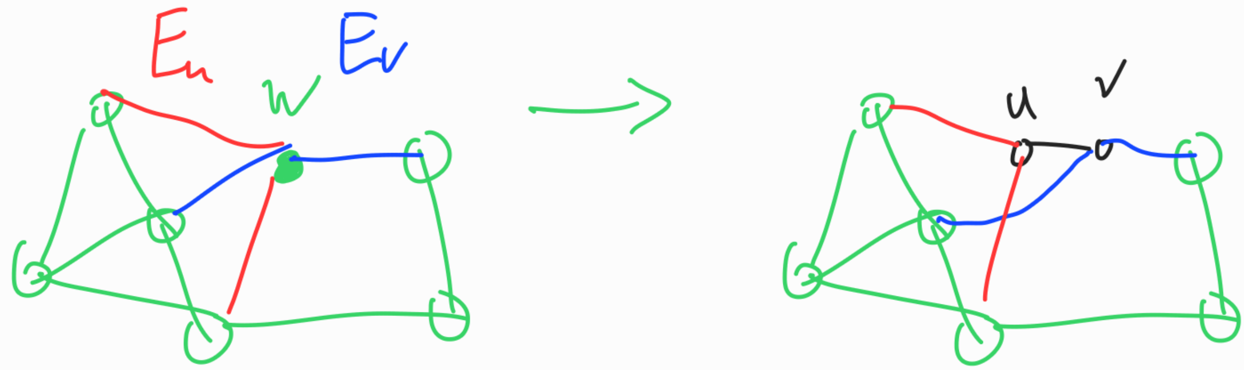
v ——— E_v,

and then adding an edge between u and v.

e.g.



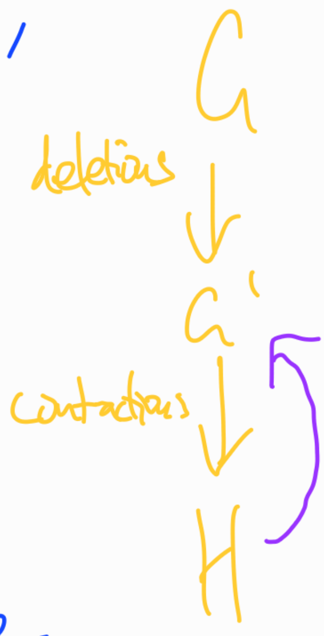
e.g.

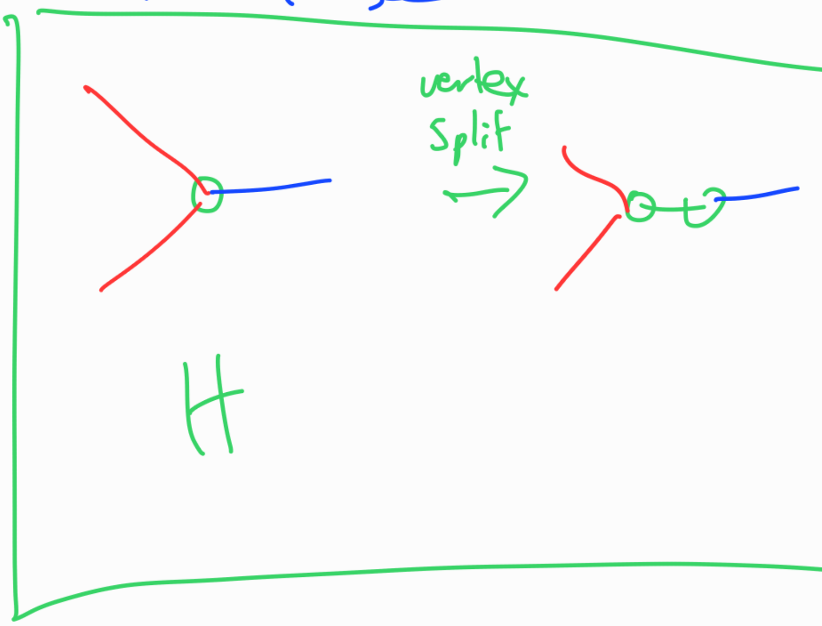


Observation: Suppose G is a graph and $e \in E(G)$. Let $H = G/e$ where w is the vertex resulting from the contraction of e. Then G can be obtained from H by a vertex split at w.

Theorem 5.19: Let H be a cubic graph and let G be a graph that has H as a minor. Then G has H as a topological minor.

Proof: Since H is a minor of G, there is a subgraph G' of G such that $H = G'/X$ for some set of edges X. We claim that G' is a subdivision of H. To see this: since $H = G'/X$, the previous observation implies that, starting from H, and performing a sequence of vertex splits, we can obtain G'.





For a subcubic graph, each vertex split at a vertex w partitions the edges incident to w into $\{E_u, E_v\}$ where one of these sets has size one. Therefore, this vertex split is a subdivision. Moreover, the graph remains subcubic. Therefore G' is a subdivision of H, as req'd. $\square$

Corollary 5.20: A graph G has a $K_{3,3}$-minor if and only if G has $K_{3,3}$ as a topological minor.