MATH 361 | Lecture 27

Assignment 4 due tomorrow

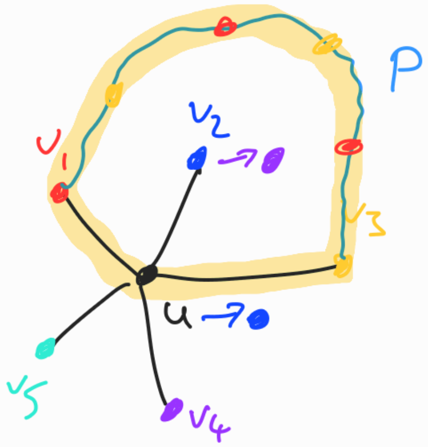
Course/teaching surveys

Last time: we were partway through proving the 5-colour theorem...

Theorem 6.12: Every loopless planar graph is 5-colourable

Recall: · induction on $|V(G)|$

· there is a degree 5 vertex u in G where $G-u$ is 5-colourable



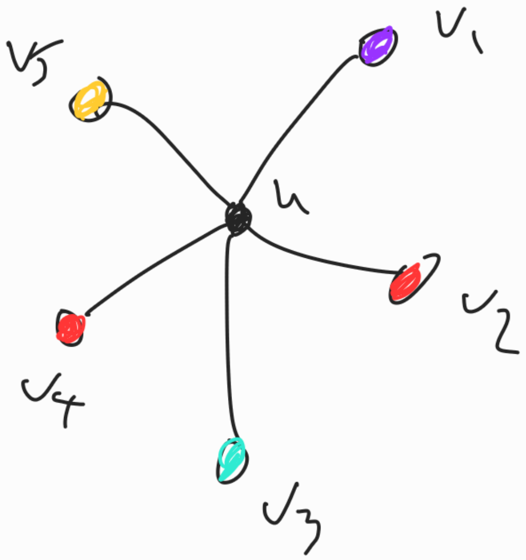
Using a R-Y Kempe chain at v_1, we saw that G is 5-colourable unless this recolours v_3 to R.

In this case, by Lemma 6.13(ii), there is a (v_1, v_3)-path P where the vertices are (alternately) R and Y.

Observe that by appending the v_1, u, v_3 path to P, we obtain a cycle C of G, where one of v_2 and v_4 is in the interior of C, and the other is in the exterior. Thus any (v_2, v_4)-path in $G-u$ must use a vertex of P. We now recolor $G-u$ using a B-M Kempe chain at v_2. Note there is no (v_2, v_4) path consisting of only B and M vertices, since such a path must contain a vertex of P. Thus v_4 remains M after the recoloring.

Now v_2 and v_4 are M, so u can be coloured B. So G is 5-colourable. $\square$

Recall that Kempe thought he had a similar argument to prove the 4-CT. Where did he go wrong? Taking a similar approach, the following situation arises (with palette {R, Y, C, M})



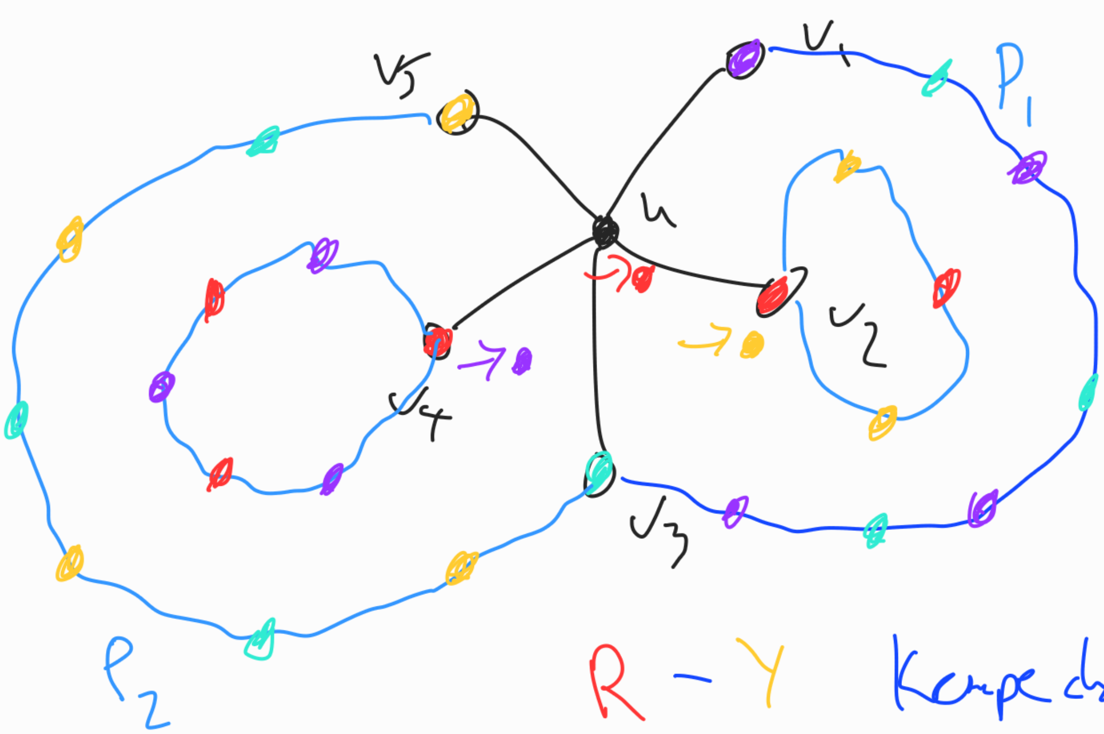
Kempe's argument:

M – C Kempe chain at v_1

→ bad case leads to path P_1

Y – C Kempe chain at v_5

→ bad case leads to path P_2

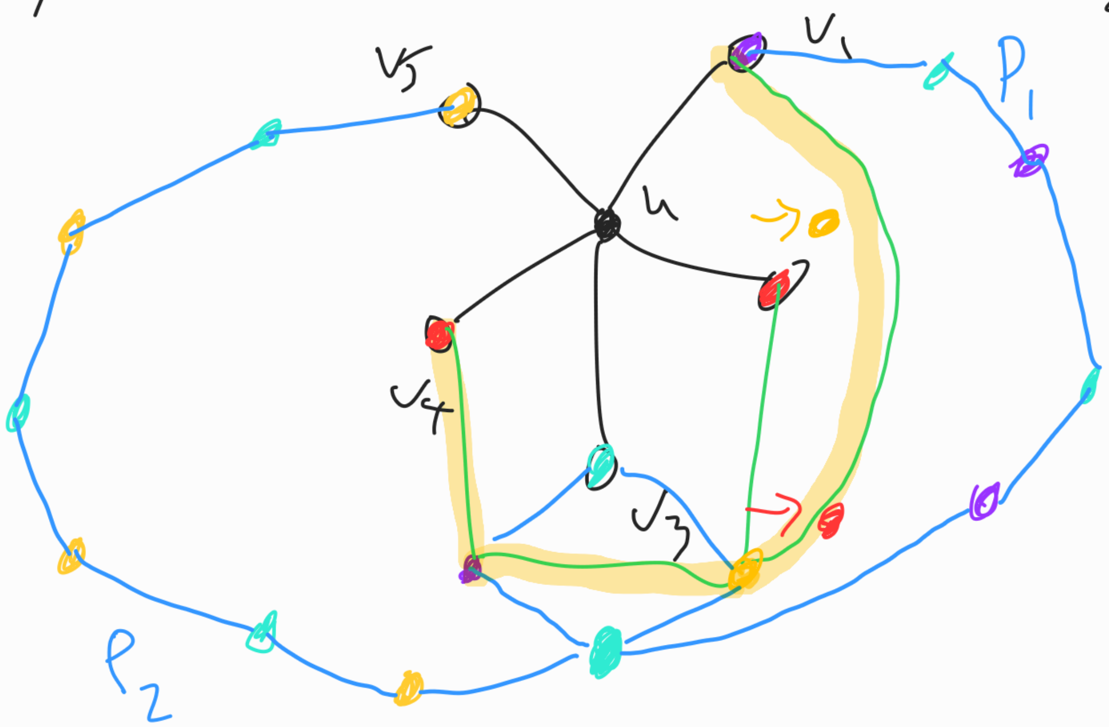


R – Y Kempe chain at v_3

R – M Kempe chain at v_4

Then we use colour R for u.

However, we could have the following problematic scenario.



What about Tait's (failed) attempt?

Tait reframed the 4-CT as an edge-colouring problem.

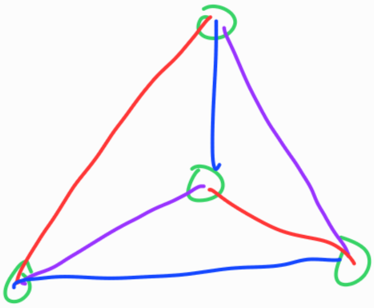
An <u>k-edge colouring</u> is a function $\varphi : E(G) \to S$

$|S| = k$.

It is <u>proper</u> if for any two adjacent edges e_1 and e_2 we have $\varphi(e_1) \neq \varphi(e_2)$

A graph is <u>k-edge-colourable</u> if it has a proper k-edge colouring.

e.g.



This illustrates a proper 3-edge-colouring of K_4.

K_4 is 3-edge-colourable.

Tait (correctly) proved:

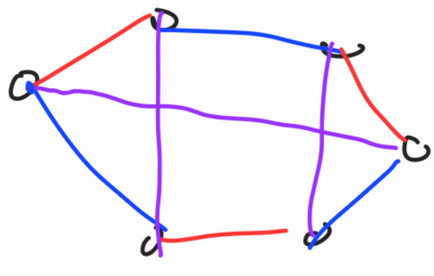
Theorem 6.15: Every loopless planar graph is 4-colourable if and only if every 3-connected cubic planar graph is 3-edge-colourable.

Recall: a Hamiltonian cycle in G is a cycle that contains all the vertices of G.

Tait observed:

Lemma 6.16: If a cubic graph has a Hamiltonian cycle, then it is 3-edge colourable.

Proof: Let $G=(V,E)$ be a cubic graph with a Hamiltonian cycle C.



By the Handshaking Lemma $3|V| = 2|E|$, so $3|V|$ is even, and hence $|V|$ is even, therefore C has even length. Now, there is a proper 2-edge-colouring of C using R and B say. Since C is a Hamiltonian cycle, every vertex of G is incident with one R edge and one B edge, so we can colour the other edge incident with the vertex M. This gives us a proper 3-edge-colouring of G. $\square$

Tait conjectured that every 3-connected cubic planar graph has a Hamiltonian cycle.

Tutte found a counterexample in 1946.