MATH 361 | Lecture 28

Teaching/course surveys - fill in online

Recap: 4-color theorem (4CT)

- Kempe's erroneous proof
- Tait's unsuccessful approach

Theorem 6.15: Every loopless planar graph is 4-colorable if and only if every 3-connected cubic planar graph is 3-edge-colorable.

Let G be a plane graph. A proper k-face colouring is a function

$$\psi : F(G) \to S \quad \text{where } |S| = k \text{ and}$$

$$\psi(f) \neq \psi(f') \text{ when } f \text{ is adjacent to } f'.$$

G is k-face-colourable if it has a proper k-face-coloring.

Theorem 6.17: A 3-connected cubic planar graph is 4-face-colourable if and only if it is 3-edge-colourable.

Proof of ($\Rightarrow$) Let G be a 3-connected cubic plane graph that has a proper 4-face-colouring $\psi : F(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$.

We define an edge-coloring φ.

For an edge e that is incident to faces f_1 and f_2, let

$$\varphi(e) = \psi(f_1) + \psi(f_2).$$

$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

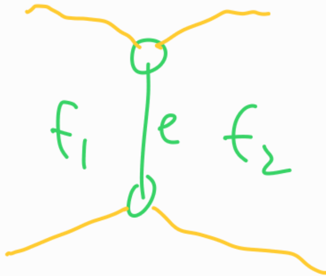
forms group with addition (coordinate-wise mod 2)

identity is $(0,0)$

Let $a, b \in \mathbb{Z}_2 \times \mathbb{Z}_2$

$a + b = (0,0)$ iff $a = b$ (*)

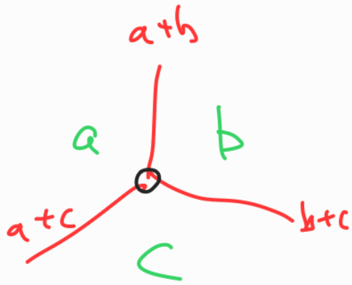
Then $\psi(f_1) \neq \psi(f_2)$, so $\varphi(e) \neq (0,0)$.



So $\varphi : E(G) \to (\mathbb{Z}_2 \times \mathbb{Z}_2) \setminus \{(0,0)\}$.

i.e. φ is a 3-edge-colouring.

It remains to show φ is proper.

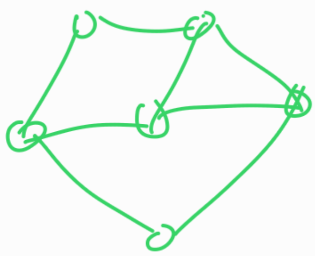


Since G is cubic, there are 3 faces incident with each vertex. Say the (distinct) colours these are assigned are a, b, c.

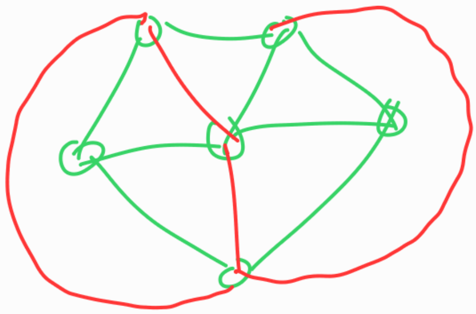
Since $\left\{\begin{matrix} a \neq c \\ b \neq c \\ a \neq b \end{matrix}\right\}$ we have $\left\{\begin{matrix} a+b \neq c+b \\ b+a \neq c+a \\ a+c \neq b+c \end{matrix}\right\}$, so $a+b$, $a+c$, and $b+c$ are distinct. $\square$

A plane graph is a <u>triangulation</u> if every face boundary, including the outer face boundary, is a cycle of length 3 (a <u>triangle</u>).

e.g.



A plane graph G



A triangulation H where G can be obtained from H by deleting the red edges.

Lemma 6.18 Every loopless planar graph is 4-colourable if and only if every triangulation is 4-colourable.

Let G be a connected plane graph.

Lemma 6.19: If G is a triangulation, then G^* is cubic.

Corollary 6.20: If G is cubic, then G^* is a triangulation.

> **Theorem 6.15:** Every loopless planar graph is 4-colourable if and only if every 3-connected cubic planar graph is 3-edge-colourable.

Proof of Thm 6.15 $(\Rightarrow)$

Suppose every loopless planar graph is 4-colourable.

Then every triangulation is 4-colourable by Lemma 6.18.

For every connected cubic plane graph G,

G^* is a triangulation (Corollary 6.20).

Since the dual of every connected cubic plane graph G is 4-colourable, every connected cubic plane graph G is 4-face-colourable (by planar duality).

In particular, every 3-connected cubic plane graph is 4-face-colorable, so, by Thm 6.17, is also 3-edge-colorable, as required.

The reverse direction is similar. □

Hadwiger's conjecture

A graph G is <u>k-chromatic</u> if its chromatic number is at least k.

k-chromatic: $\chi(G) \geq k$
k-colorable: $\chi(G) \leq k$

<u>Conjecture</u> 6.23 (Hadwiger's Conjecture, 1943)
Let G be a loopless graph, and let k be a positive integer. If G is k-chromatic, then G has a K_k-minor.

The cases $k \leq 4$ are not too difficult.

$k=5$ was shown to be equivalent to 4CT
$k=6$ shown by Robertson, Seymour, Thomas (1993)
$k \geq 7$ open.