MATH361 | Lecture 4

Assignment 1 due Mon 11 March 3pm

Last time:
- cycles, complete graphs, bipartite graphs
- graph isomorphism
- trees and forests
- bridge

Thm 2.1: Let G be a connected graph, with $e \in E(G)$. The edge e is a bridge if and only if ~~e~~ it is not in any cycle of G.

Corollary 2.2: Let G be a graph. G is a tree if and only if G is connected and every edge of G is a bridge.

This is a characterization of trees.

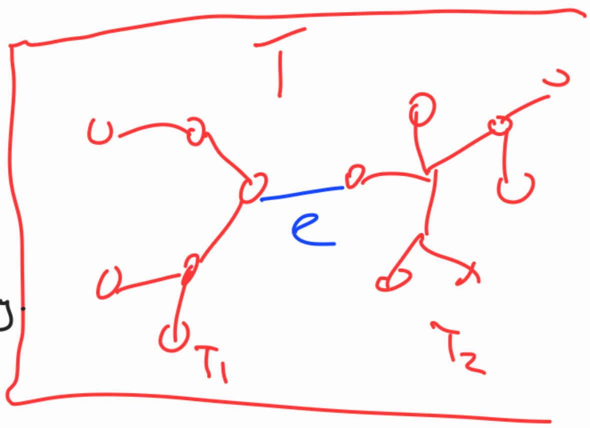
When G is connected and e is a bridge, $G \setminus e$ has at least 2 components. In fact...

Lemma 2.4: If G is a connected graph and e is a bridge of G then $G \setminus e$ has precisely 2 components.

Theorem 2.5: Let T be a tree with n vertices, with $n \geq 1$. Then T has $n-1$ edges.

Proof: By strong induction on n. If $n=1$, T certainly has 0 edges, so the result holds in this case.

Assume $n > 1$, and that the result holds for any non-empty tree with fewer than n vertices. Observe that T has at least one edge, since $n > 1$. Let e be an edge of T. By Corollary 2.2 e is a bridge, so, by Lemma 2.4 $T \setminus e$ has 2 components, T_1 and T_2 say.



As T_1 and T_2 are connected, and have no cycles (since they are subgraphs of T, which has no cycles), they are trees, so by the induction assumption

$$|E(T_1)| = |V(T_1)| - 1 \quad \text{and} \quad |E(T_2)| = |V(T_2)| - 1.$$

Now
$$\begin{aligned} |E(T)| &= |E(T_1)| + |E(T_2)| + 1 \\ &= (|V(T_1)| - 1) + (|V(T_2)| - 1) + 1 \\ &= |V(T_1)| + |V(T_2)| - 1 \\ &= n - 1 \qquad \text{since } |V(T)| = |V(T_1)| + |V(T_2)| \end{aligned}$$

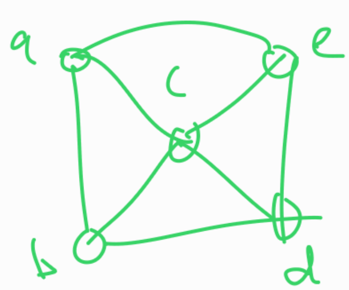
The result follows by strong induction. $\square$

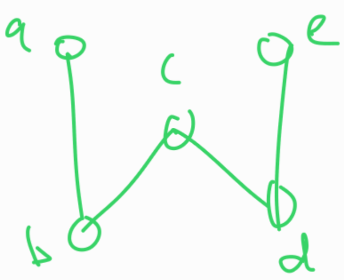
Spanning trees

Let G be a graph. A spanning tree of G is a subgraph H of G such that H is a tree and $V(H) = V(G)$.

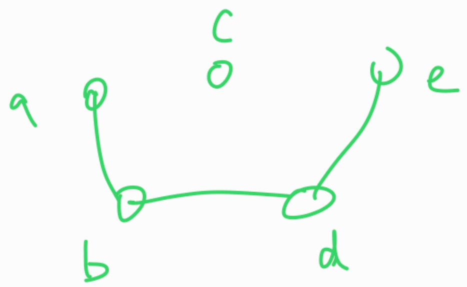
G $\qquad\qquad$ H

e.g.

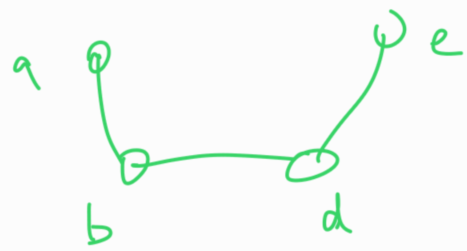




H is a spanning tree of G.



and



are <u>not</u> spanning trees of G.

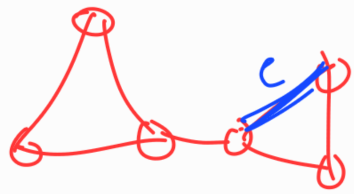
<u>Thm</u> 2.8: Let G be a graph

G is connected if and only if G has a spanning tree.

<u>Proof</u>: ($\Leftarrow$) Suppose G has a spanning tree H. For every pair of vertices $\{u,v\}$ in H, there is a path from u to v (since H is a tree and is therefore connected). But any such path is also a path in G. Since $V(H) = V(G)$ there is a path between every pair of vertices in G, so G is connected.

($\Rightarrow$) Suppose G is connected.

If G is a tree, then G itself is a spanning tree of G, as required. Suppose G is not a tree. Then G has an edge e that is not a bridge (by Corollary 2.2). Then



$G \setminus e$ is connected (by the definition of a bridge) and has the same vertex set as G. We can iteratively repeat this process, always maintaining a graph on vertex set $V(G)$. Eventually, we obtain a connected graph with only bridges, on vertex set $V(G)$. By Corollary 2.2, this graph is a tree, so it is a spanning tree of G. $\square$

<u>Corollary</u> 2.9: Let G be a non-empty graph. G is a tree if and only if G is connected and has $|V(G)|-1$ edges.