MATH361 | Lecture 5

Assignment 1 due
3pm Mon 11 March

Yesterday: trees and spanning trees

Thm 2.5: A non-empty tree with n vertices has $n-1$ edges

More generally:

Thm 2.6: A non-empty forest with n vertices and c components has $n-c$ edges.

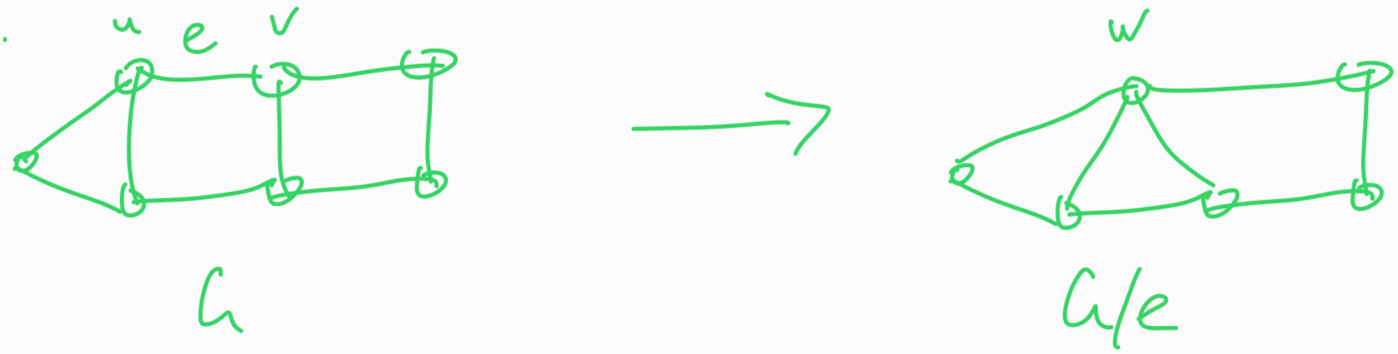
Exercise 2.7: prove this.

Contraction

Let G be a graph.

Intuitively, the contraction of an edge $e \in E(G)$, denoted G/e, "shrinks" it down to a single vertex

e.g.



More formally:

e.g.

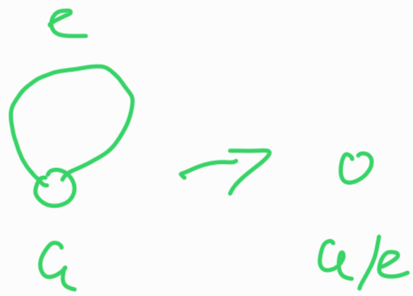
for a non-loop edge $e = uv$

$$E(G/e) = E(G) \setminus \{e\}$$

$$V(G/e) = (V(G) \setminus \{u,v\}) \cup \{w\}$$

where w is a vertex not in $V(G) \setminus \{u, v\}$

and each edge in $E(G) \setminus \{e\}$ is incident with the same vertices as in G except if incident with u or v, in which case the incidence with u or v is replaced with w in G/e.



How about loops?

For a loop e, we define

$$G/e = G \setminus e.$$

Note: the notation G/e is useful, but it doesn't specify what label is given to the vertex resulting from the contraction.

What about deleting vertices?

Let $v \in V(G)$ and $X \subseteq V(G)$

We let $\begin{cases} G - v \\ G - X \end{cases}$ denote the graph obtained from G

by removing the $\begin{cases} \text{vertex } v \\ \text{vertices } X \end{cases}$ and all edges incident with $\begin{cases} v \\ X \end{cases}$.

Subgraphs can be obtained by deleting edges and vertices,

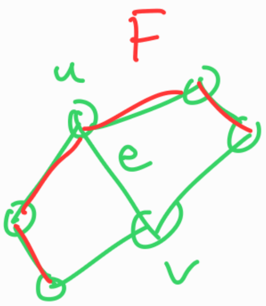
Induced subgraphs can be obtained by deleting vertices only.

A graph H is a minor of a graph G if H can be obtained from G by a (possibly empty) sequence of edge deletions, edge contractions and vertex deletions.

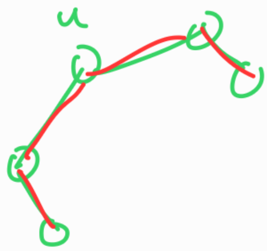
Lemma 1.9(ii) Let G be a graph with a non-loop edge $e = uv$, and $F \subseteq E(G)$ such that F is incident with at most one of $\{u, v\}$.

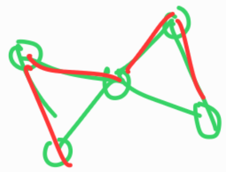
Then $\quad G/e[F] \cong G[F]$

e.g.



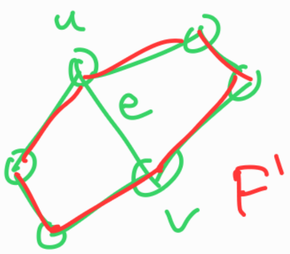
G

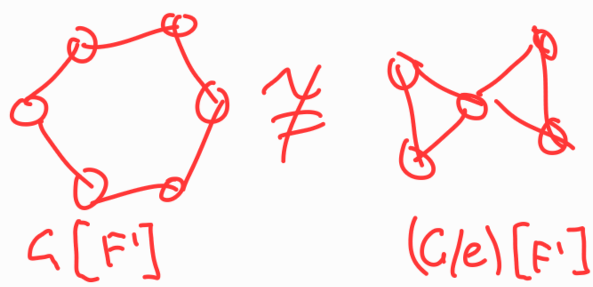




G/e

$G[F] \cong G/e[F]$

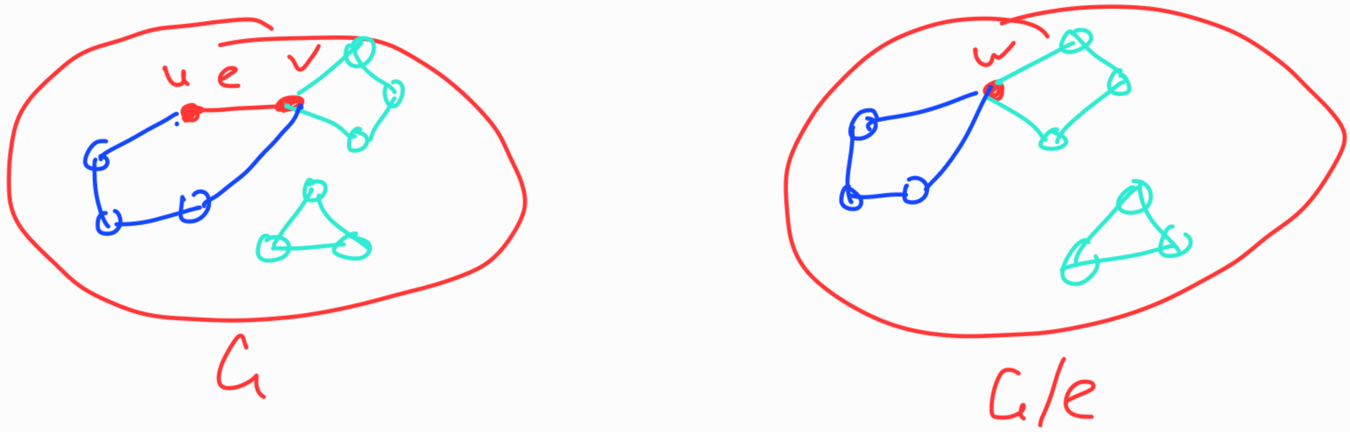




Lemma 1.12: Let G be a graph with an edge $e=uv$ and let C be a non-empty subset of $E(G)\setminus\{e\}$. Then C is the set of edges of a cycle in G/e if and only if either

(i) $C\cup\{e\}$ is the set of edges of a cycle in G, or

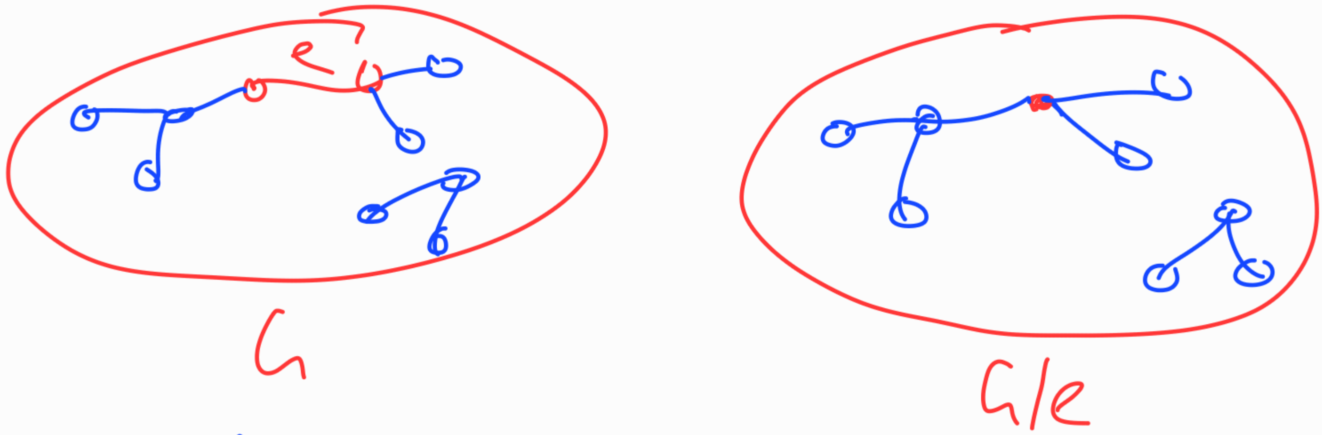
(ii) C is the set of edges of a cycle in G and the vertices of this cycle contain at most one of $\{u,v\}$.



Lemma 2.12

Let G be a graph with a non-loop edge e and $F \subseteq E(G)\setminus\{e\}$.

F is the edge set of a forest in G/e if and only if $F\cup e$ is the edge set of a forest in G.



<u>Proof of</u> of $(\Leftarrow)$ (via the contrapositive).

Say $(G/e)[F]$ is not a forest.

Then $(G/e)[F]$ contains a cycle.

Let $C \subseteq F$ be the edges of this cycle in G/e.

Then, by Lemma 1.12, either $G[C]$ or $G[C \cup \{e\}]$ is a cycle. Since

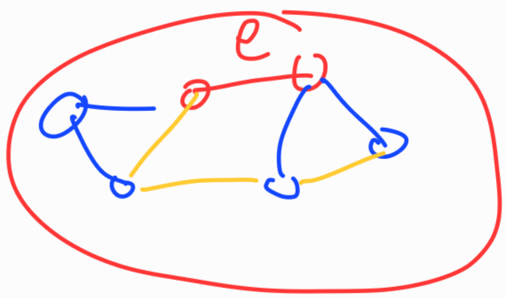
$$C \cup \{e\} \subseteq F \cup \{e\},$$

$G[F \cup \{e\}]$ is not a forest.

$(\Rightarrow)$ in online notes.

<u>Lemma 2.13</u>: Let G be a connected graph with a non-loop edge e, and let $F \subseteq E(G) \setminus \{e\}$. Then F is the edge set of a spanning tree in G/e

if and only if $F \cup \{e\}$ is the edge set of a spanning tree in G.



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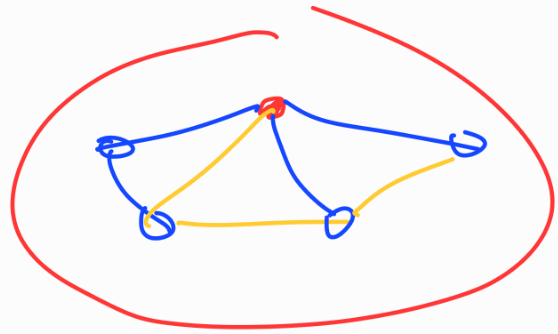

G/e