Assign 2 out now

Last time:

Thm 3.4: Let G be a 2-connected graph with $|V(G)| \geq 4$ and $e \in E(G)$. Then at least one of $G \setminus e$ and G/e is 2-connected.

An isolated vertex in a graph is a vertex with degree 0.

Thm 3.7: Let G be a graph with no isolated vertices, and no loops and $|V(G)| \geq 3$.

G is 2-connected if and only if for every pair $\{e,f\} \subseteq E(G)$ there is a cycle whose edge set contains $\{e,f\}$.

We'll need:

Lemma 3.5: Let G be a 2-connected graph with parallel edges $\{e,f\}$. Then $G \setminus e$ is 2-connected

Proof of ($\Rightarrow$) of 3.7:

Suppose G is 2-connected. Then $|V(G)| \geq 3$ and G is connected. Since G has no cut vertices, $|E(G)| \geq 3$.

Base case:

If $|E(G)| = 3$, then $G \cong K_3$, and every pair of edges of K_3 is contained in a cycle.

Induction assumption:

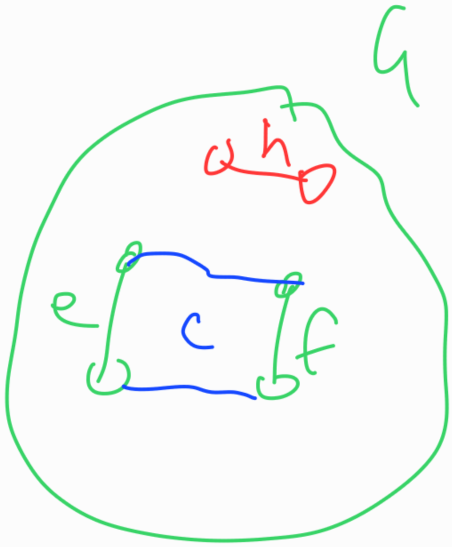
Now suppose $|E(G)| \geq 4$, and assume the ($\Rightarrow$) direction holds for any graph with $|E(G)|-1$ edges.

Let e and f be arbitrary edges in G. We want to show that there is a cycle of G containing $\{e,f\}$.

Let $h \in E(G) \setminus \{e,f\}$. By Thm 3.4, if $G \setminus h$ is not 2-connected, then G/h is 2-connected.

Case ①: $G \setminus h$ is 2-connected.

Then $G \setminus h$ has a cycle C containing $\{e,f\}$ by the induction assumption, and C is also a cycle of G.



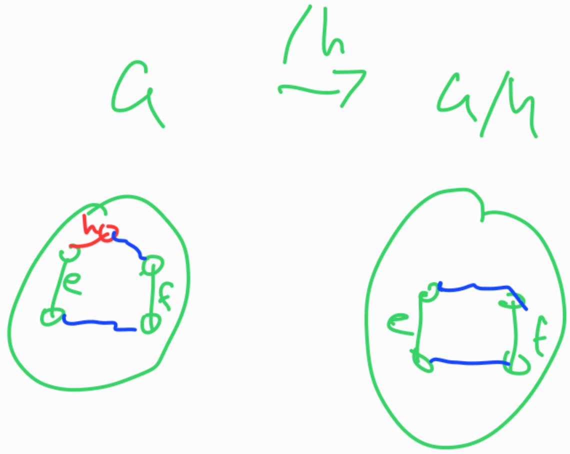
Case ②: $G \setminus h$ is not 2-connected.

So G/h is 2-connected.

If G/h has a loop, then h is in a parallel pair in G. But then $G \setminus h$ is 2-connected

by Lemma 3.5, contradicting that we're in case 2.

Now G/h is loopless and 2-connected. By the induction assumption G/h has a cycle C containing $\{e, f\}$.



Then, by Lemma 1.12, either C is a cycle in G or there is a cycle in G on edge set $E(C) \cup \{h\}$. In either case, we have a cycle in G containing $\{e, f\}$. The result follows by induction. $\square$

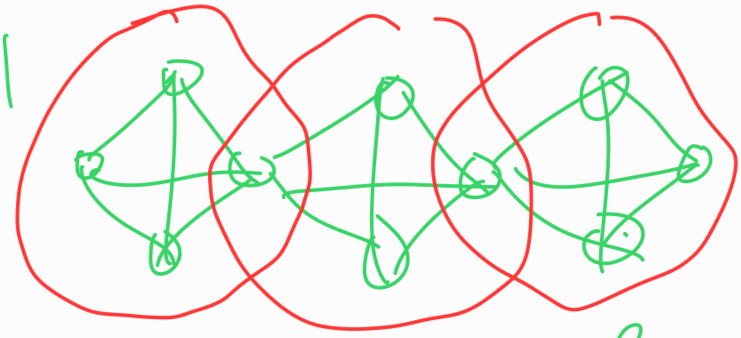
($\Leftarrow$) is a tutorial question.

<u>Corollary</u> 3.8: Let G be a 2-connected graph and $u, v \in V(G)$. Then there is a cycle containg both u and v.

Blocks

Recall that a component is a maximal connected subgraph

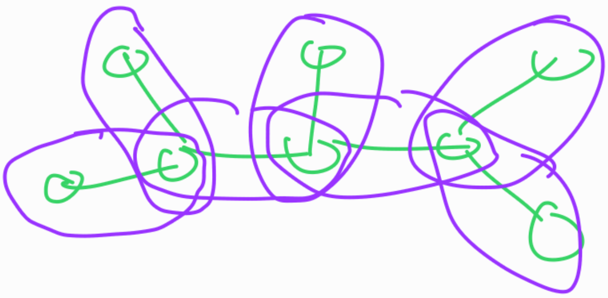
e.g. 1



G.

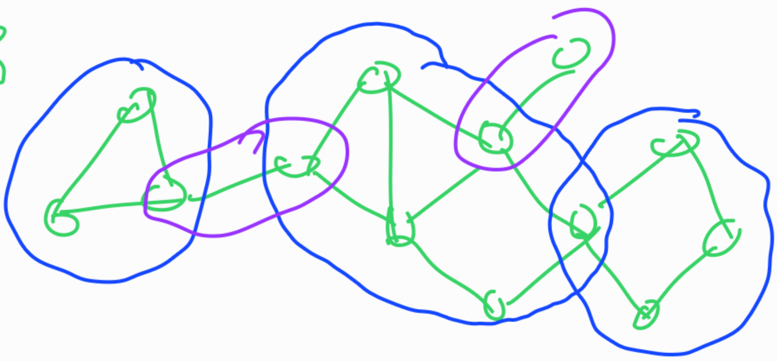
these are maximal 2-conn'd subgraphs of G.

e.g. 2



bridges

e.g. 3



maximal 2-conn'd subgraphs

Let G be a loopless graph.

We say G is biconnected if it is connected and has no cut vertices.

A block of G is a maximal biconnected subgraph of G

Some properties of blocks in a graph G

* two distinct blocks have at most one vertex in common (which will be a cut vertex of G).

* each edge of G belongs to exactly one block.