MATH361 | Lecture 9

Recap: Thm 3.7: Let G be a connected graph with no isolated vertices or loops, and $|V(G)| \geq 3$.

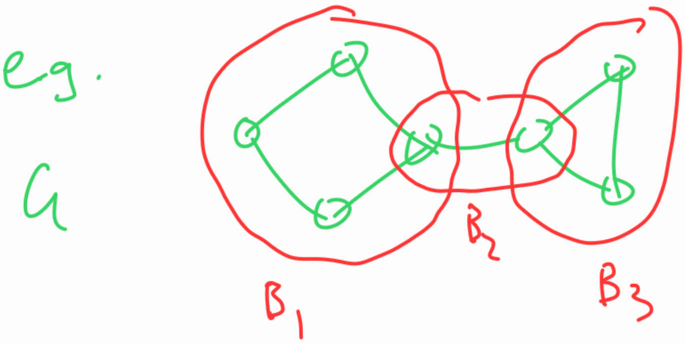
G is 2-connected if and only if for every pair $\{e,f\} \subseteq E(G)$, there is a cycle containing $\{e,f\}$.

→ **blocks**

For a loopless graph G, a block is a maximal biconnected subgraph

a graph is biconnected if it is connected and has no cut vertices

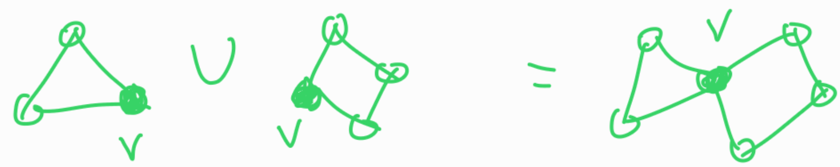
e.g.



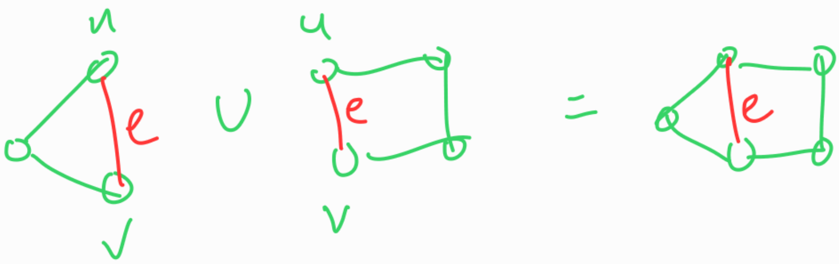
G has 3 blocks: B_1, B_2 and B_3.

Graph union

e.g. 1



e.g. 2



In e.g. 2, we require the vertices incident with e to be "consistent".

We say graphs G_1 and G_2 are consistent if for every $e \in E(G_1) \cap E(G_2)$, the vertices incident with e in G_1 are the vertices incident with e in G_2.

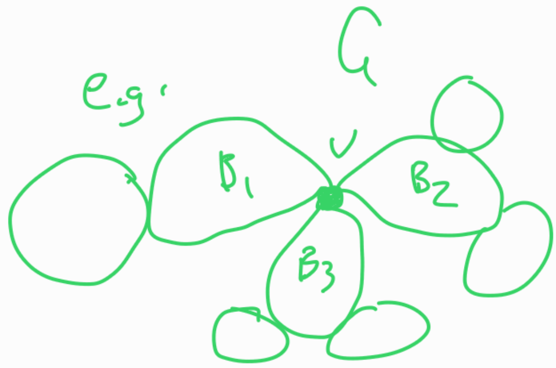
For consistent graphs G_1 and G_2, we define the union of G_1 and G_2 to be the graph on vertex set $V(G_1) \cup V(G_2)$ and on edge set $E(G_1) \cup E(G_2)$ (with incidences inherited from G_1 and G_2). We denote this graph by $G_1 \cup G_2$.

Exercise: Prove that if G_1 and G_2 are consistent connected graphs with a vertex in common, then $G_1 \cup G_2$ is connected.

i.e $|V(G_1) \cap V(G_2)| \geq 1$

Back to blocks:

Lemma 3.12: Let G be a loopless connected graph. If v is a cut vertex of G, then v belongs to at least two distinct blocks of G.



↓

block-cut graph of G. $\mathcal{B}(G)$

Let G be a loopless graph,

where $\mathcal{B} = \{B_1, B_2, B_3, \dots, B_t\}$

is the set of blocks of G,

and S is the set of cut vertices of G.

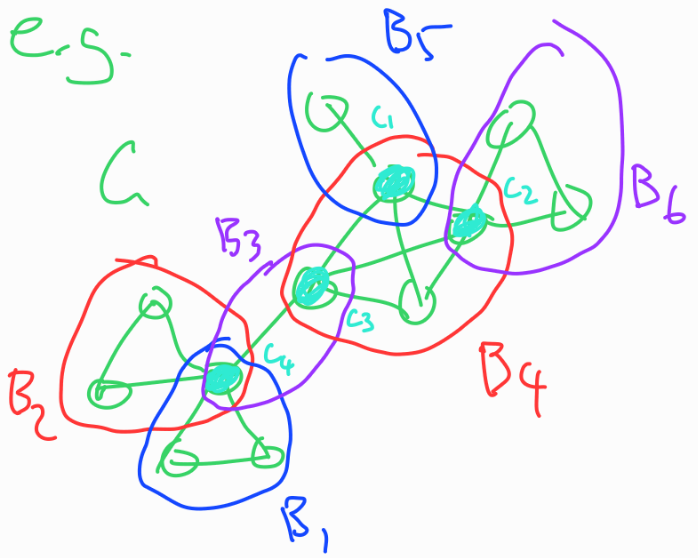
Then we can define the block-cut graph $B(G)$ of G

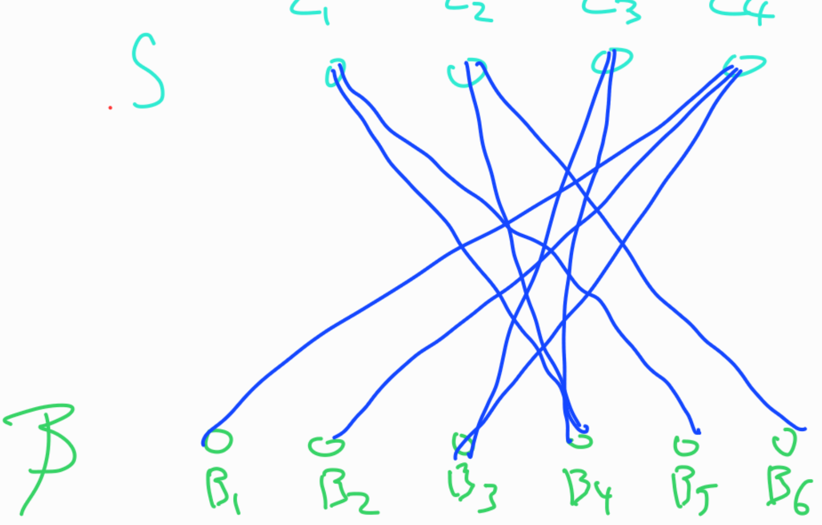
to be the bipartite graph with bipartition

$\{\mathcal{B}, S\}$, such that there is an edge between

$B \in \mathcal{B}$ and $v \in S$ if and only if the block B

contains v.





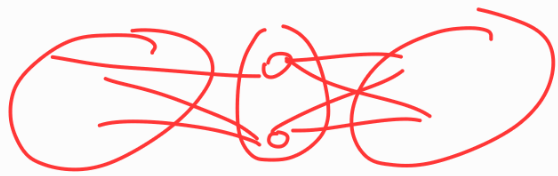
Note: by Lemma 3.12, each vertex in S has degree at least 2.

Lemma 3.14 (i): Let G be a loopless graph. B(G) is a forest.

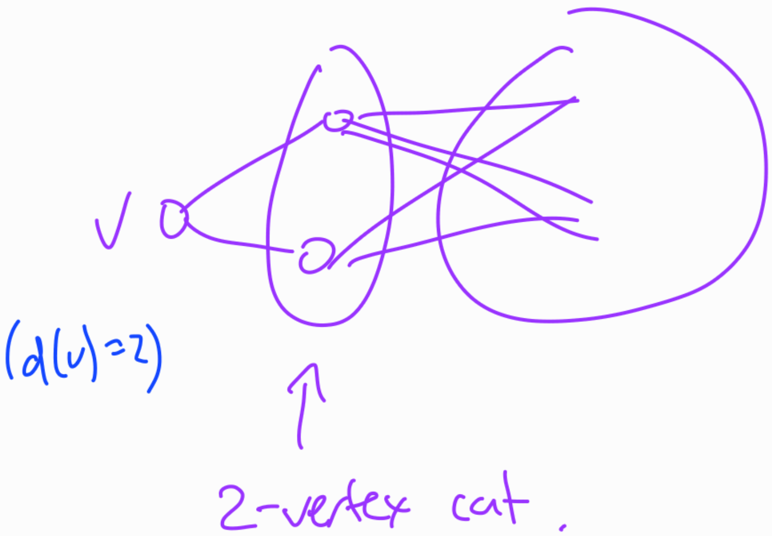
3-connected graphs

Recall a 3-connected graph G has no 1- or 2-vertex cuts, and G has at least 4 vertices and is connected.

Note: a 3-connected graph has no vertices of degree ≤ 2.

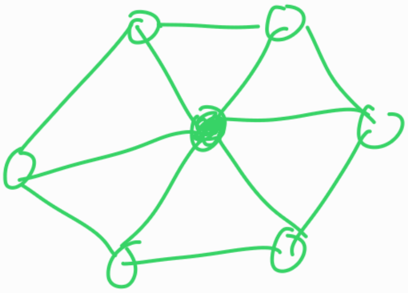


2-vertex cut



Let G be a 3-connected graph.

(1) Does G always have some edge e such that $G \setminus e$ is 3-connected? No



This graph is 3-connected

However, for every edge e $G \backslash e$ is not 3-connected (since it has a vertex of degree 2)

(2) For every edge of a 3-connected graph G, is either G/e or $G \backslash e$ 3-connected?

Still no.

not 3-connected.

However, G/f is 3-connected here.

<u>Theorem</u> 3.21: Let G be a 3-connected graph with $|V(G)| \geq 5$. Then there exists an edge $e \in E(G)$ such that G/e is 3-connected.