4. Planar Graphs I: Basics and Planar Duals

Let H be a graph. Informally, we are interested in the question: can H be drawn in the plane (i.e. on a piece of paper) such that no pair of edges cross?

To describe this more formally, we define the notion of a planar embedding as follows. A *planar embedding* of H is a pair of functions (f, g) where f maps each vertex of H to a point in the plane, and g maps each edge of H to a simple curve in the plane, such that:

(i) for an edge e of H with ends u and v, the curve $g(e)$ is from the point $f(u)$ to the point $f(v)$, and
(ii) the only points on the plane where two distinct curves $g(e_1)$ and $g(e_2)$ meet is at a point $f(u)$ for some vertex u of H.

A *plane graph* is a graph together with a planar embedding of the graph.

> Note that we can have a planar embedding of a graph with loops, where the embedding maps a loop edge to a simple curve that starts and ends at the same point.

A plane graph is a graph with extra structure (that describes how to draw the graph in the plane without crossing edges).

If we have a plane graph G, then the graph we obtain by ignoring the planar embedding is called the *underlying graph* of G. For simplicity, when we refer to edges/vertices/cycles (and so on) of a plane graph G, we are really referring to edges/vertices/cycles (and so on) of the underlying graph of G.

A *planar graph* is a graph that has a planar embedding. In other words, a graph is planar if it is the underlying graph of some plane graph. In particular, if G is a plane graph, then the underlying graph of G is planar.

Suppose G is a plane graph, with underlying graph H and embedding (f, g). Often we will be interested in the points or curves that vertices or edges of H are mapped to by f or g respectively. For simplicity, when we refer to the embedding of a set of vertices or edges in a plane graph, we mean the points or curves that these vertices or edges are mapped to by f or g respectively.

Topological Issues. Discussion of plane graphs necessarily involves the study of the topology of the plane. To do so in a rigorous manner is well beyond the scope of the course.

A *curve* is a continuous image of a closed unit line segment. A *closed curve* is a continuous image of a circle. A curve or a closed curve is *simple* if it does not intersect itself. The next elementary fact is fundamental.

Lemma 4.1. *The edges of any cycle in a plane graph form a simple closed curve.*

A subset X of the plane is *arcwise-connected* if any two points in X can be connected by a curve contained entirely in X. The next result is one of the most fundamental theorems of topology.

Theorem 4.2 (The Jordan Curve Theorem)**.** *Any simple closed curve in the plane partitions the rest of the plane into two disjoint arcwise-connected open sets.*

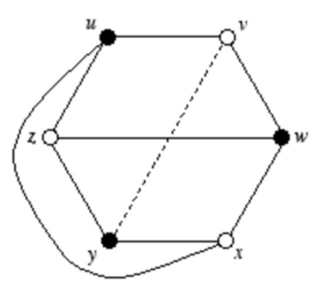
The Jordan Curve Theorem may seem obvious, but in fact it is quite tricky to prove. Indeed, the first person to notice that it needed a proof was Bolzano in the early 1800s.

The two sets into which a simple closed curve C partitions the rest of the plane are called the *interior* and the *exterior*, and are denoted by $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ respectively. We let $\mathrm{Int}(C) = \mathrm{int}(C) \cup C$, and let $\mathrm{Ext}(C) = \mathrm{ext}(C) \cup C$ (topologically, $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ are the *closures* of $\mathrm{int}(C)$ and $\mathrm{ext}(C)$, respectively). Observe that $\mathrm{Int}(C) \cap \mathrm{Ext}(C) = C$.

To prove that a graph *is* planar, it suffices to give a planar embedding. But it's a different issue proving that a graph *is not* planar. We'll develop tools for doing this. However, with a careful use of The Jordan Curve Theorem and Lemma 4.1, in some cases it is possible to do this in a way that might even keep the more pedantically minded happy.

Theorem 4.3. *$K_{3,3}$ is non-planar.*

Proof. Consider $K_{3,3}$, with the vertices labelled as in the following drawing.



The outer cycle along the vertices u, v, w, x, y, z, u is a *Hamiltonian cycle* of $K_{3,3}$, that is, it is a cycle on vertex set $V(K_{3,3})$. By Lemma 4.1, in any planar embedding, the edges of a cycle must be a simple closed curve. Suppose there is a planar embedding of $K_{3,3}$, and let C be the simple closed curve for this Hamiltonian cycle. By the Jordan Curve Theorem, C partitions the rest of the plane into two arcwise-connected regions, $\mathrm{int}(C)$ and $\mathrm{ext}(C)$. By the definition of a planar embedding, the edges ux, vy, and wz must lie entirely in either $\mathrm{int}(C)$ or $\mathrm{ext}(C)$. So (at least) two of these edges lie in one of these regions. But then these two edges must cross. From this contradiction, we deduce that there is no planar embedding of $K_{3,3}$; that is, $K_{3,3}$ is not planar. □

What about K_5? Some of you might have seen in MATH161 a proof that K_5 is non-planar using a corollary of Euler's Formula. We haven't seen these tools yet in MATH361.

Exercise 4.4. *Prove K_5 is non-planar (without using Euler's formula).*

Subdivisions: Let G be a graph with an edge $e = uv$. Let H be the graph obtained from G by removing the edge e, adding a new vertex w, and adding edges $e' = uw$ and $e'' = wv$. We call this operation an *edge subdivision*, and say that H is obtained from G by *subdividing* e. Note that w has degree two in H, while all all other vertices have the same degree as in G.

→ Despite the slightly unwieldy definition, an edge subdivision is really an elementary operation: intuitively, we are just putting a new vertex "on" an existing edge.

Lemma 4.5. *If H is obtained from a graph G by subdividing the edge $e \in E(G)$, with new edges e' and e'', then $G \cong H/e' \cong H/e''$.*

If the graph H is obtained from G by a sequence of edge subdivisions, then we say that H is a *subdivision* of G.

Lemma 4.6. *Let H be a subdivision of a graph G. Then H is planar if and only if G is planar.*

Spherical Embeddings: As well as drawing graphs in the plane we can draw them in other surfaces, for example, a Möbius band, the torus, or the Klein bottle. The study of such embeddings is the beautiful branch of mathematics known called *topological graph theory.* Unfortunately, to take that mathematical journey we really do need to treat the topology seriously and that makes much of it beyond the scope of MATH361. However, one surface that is of particular interest is the surface of the 3-dimensional sphere. It turns out that the graphs embeddable on this surface belong to a familiar class.

Theorem 4.7. *A graph is embeddable on the plane if and only if it is embeddable on the sphere.*

It is not hard to see intuitively why Theorem 4.7 is true. A formal proof using projective geometry is also not too difficult. One advantage of spherical embeddings is that they are more symmetric than planar embeddings as there is no special “outer face”.

Faces: Let G be a plane graph. Since the embedding of the edges of a cycle of G is a simple closed curve, they partition the rest of the plane into two arcwise-connected open sets (the interior and the exterior of the curve). It follows that, considering the embedding of all of the edges of G, we obtain a partition of the rest of the plane into a collection of arcwise-connected open sets, which are called the *faces* of G (or the *faces* of the planar embedding of G). We denote by $F(G)$ the set of faces of G. Each plane graph has exactly one unbounded face called the *outer face*.

The *boundary* of a face f of G is the boundary of the open set f in the topological sense. This boundary corresponds to a closed walk in the underlying graph, which we (usually) regard as a subgraph (consisting of the vertices and edges of the closed walk). We say that the face f is *incident* with the vertices and edges in this closed walk. Two distinct faces are *adjacent* if their boundaries have an edge in common. For a face f, we denote the edges in the boundary of f by $\partial(f)$.

The next theorem can be proved using essentially the same techniques as Theorem 4.7.

Theorem 4.8. *Let G be a planar graph and let f be a face in some planar embedding of G. Then there exists a planar embedding of G in which the outer face has the same boundary as f.*

We next give a few more intuitively obvious facts without proof.

Lemma 4.9. *Let G be a plane graph.*

(i) *If G is a forest, then G has exactly one face.*
(ii) *If G is connected, and f is a face of G, then $G[\partial(f)]$ is connected.*
(iii) *If e is a bridge of G, then e is incident with exactly one face; otherwise e is incident with two faces.*

Something for the Connoisseur: To prove the above things properly, we need more topology. Indeed for some of them we need the following theorem.

Theorem 4.10 (The Jordan–Schönflies Theorem)**.** *Any homeomorphism of a simple closed curve in the plane onto another simple closed curve in the plane can be extended to a homeomorphism of the plane.*

What does this say? Imagine I have two simple closed curves — which we can think of as being made of rubber. Then a homeomorphism of one to the other is a way of stretching the rubber to change one into the other. To understand what a homeomorphism of the plane is, you have to imagine the whole plane being a rubber sheet. If you can get from one curve to the other by stretching the rubber band, you can extend that to a stretch of the whole rubber sheet!

Topology is an amazing subject. It studies geometric properties that have nothing to do with distances, angles, etc. The development of topology is one of the highlights of twentieth century mathematics. There is a careful treatment of the topological aspects of planar graphs in the textbook *Graph Theory* by Diestel. The textbook *Graphs and Surfaces* by Mohar and Thomassen gives an excellent treatment of the whole of topological graph theory.

More terminology: The *degree* of a face, denoted $d(f)$, is the number of edges on its boundary (that is, the cardinality of $\partial(f)$), except we count a bridge twice.

→ You might like to think of a bridge being counted twice because, as we traverse the face boundary, we traverse each bridge twice.

For a graph G we have the following:

- $E(G)$ is the set of edges of G,
- $V(G)$ is the set of vertices of G, and
- $d(v)$ is the degree of a vertex v in G.

If G is a plane graph we also have:

- $F(G)$ is the set of faces of G, and
- $d(f)$ is the degree of a face f of G.

Whitney's Theorem. Looking at a plane graph, it is clear that the boundary of a face is typically a cycle, although that's not always the case as we've seen when the graph has bridges.

But even when there are no bridges, we can have face boundaries that are not cycles, for example if the graph has cut vertices. But Hassler Whitney, in 1932, proved that once we get to 2-connected graphs, the problem disappears, and all face boundaries are cycles. Before we get to that result, we need a lemma.

Lemma 4.11. *Let G be a plane graph, let e be an edge of G that is not a bridge or a loop. Let f_1 and f_2 be the faces of G incident with e. Then there is a planar embedding of G/e such that*

(i) *$\partial(f_1) \setminus \{e\}$ and $\partial(f_2) \setminus \{e\}$ are each the set of edges of a face boundary, and*
(ii) *$\partial(f)$ is the set of edges of a face boundary when f is a face of G such that $f \notin \{f_1, f_2\}$.*

What happens if e is a bridge of G? What happens if e is a loop of G?

The next corollary is an immediate consequence of Lemma 4.11.

Corollary 4.12. *If G is a planar graph, then any minor of G is planar.*

And now we can prove Whitney's Theorem.

Theorem 4.13 (Whitney, 1932)**.** *Let G be a loopless 2-connected plane graph. Then every face boundary of G is a cycle.*

Proof. The proof is by induction on the number of edges of G. Since G is 2-connected, it has at least three vertices, and it follows that G has at least three edges. If G has precisely three edges, then $G \cong K_3$. In this case, G is a plane graph consisting of a single cycle. By the Jordan Curve Theorem, G has two faces, and the boundary of each face is a cycle.

Now assume that G has more than three edges and that the theorem holds for loopless 2-connected plane graphs with fewer edges than G. Let $e = uv$ be an edge of G. By Theorem 3.4, either $G\backslash e$ or G/e is 2-connected. We consider two cases depending on whether or not $G\backslash e$ is 2-connected.

Assume that $G\backslash e$ is 2-connected. Consider a plane drawing of G. Deleting the edge e gives a plane drawing of $G\backslash e$. Because G is a plane graph, the vertices u and v must lie on a common face boundary, as otherwise we would not be able to insert the edge e. By the induction assumption, this face boundary is a cycle. Say that a closed walk along the cycle visits the vertices in the following sequence: $u, x_1, \ldots x_s, v, y_1, \ldots, y_t, u$. Then, since $e = uv$, we see that the two face boundaries that contain e are the cycles corresponding to the closed walks $u, x_1, \ldots x_s, u$ and $v, y_1, \ldots, y_t, v$. All other face boundaries of G are also face boundaries of $G\backslash e$, which are cycles of $G\backslash e$ by the induction assumption, and hence also cycles of G. Thus the result holds if $G\backslash e$ is 2-connected.

Now assume that $G\backslash e$ is not 2-connected, so G/e is 2-connected by Theorem 3.4. It follows, by Lemma 3.5, that e is not in a parallel pair, so G/e is loopless (using the same argument as we employed in Theorem 3.7). Hence we can apply the induction assumption on G/e. Let f_1 and f_2 be the two faces of G incident with e, and note that e is not a bridge of G (since G is 2-connected, using Lemma 2.18). Thus, there is a planar embedding of G/e as described in Lemma 4.11. By the induction assumption, every face boundary of this planar embedding of G/e is a cycle in G/e. Observe, in particular, that each face boundary of G, other than those for f_1 and f_2, contains at most one of u and v. First consider the face f_1. By Lemma 4.11(i), $\partial(f_1) \setminus \{e\}$ is the set of edges of a face boundary of G/e, so this set of edges induces a cycle of G/e. It follows, by Lemma 1.12, that $G[\partial(f_1)]$ is a cycle of G. Similarly, $G[\partial(f_2)]$ is a cycle of G. Now consider a face f of G other than f_1 or f_2. By Lemma 4.11(ii), the boundary of f in G is also a face boundary of G/e, so $G[\partial(f)]$ is a cycle in G/e. It follows, by Lemma 1.12, that $G[\partial(f)]$ is a cycle of G. □

Theorem 4.13 is a good example of how we gain structure by raising connectivity. Put another way: with low connectivity, annoying junk can appear, but we can get past the junk by raising the connectivity.

Planar Duals. Given a plane graph G, one can define another plane graph G^*, with vertex set $V(G^*) = \{f^* : f \in F(G)\}$ and edge set $E(G^*) = \{e^* : e \in E(G)\}$, as follows:

- If $e \in E(G)$ is incident to faces f and g in G, then f^* and g^* are incident with e^* in G^*.
- In the planar embedding of G^*, for each $f^* \in V(G^*)$ we choose a point in f.
- In the planar embedding of G^*, for each edge e^* with ends f^* and g^* (where $f^* = g^*$ only if e^* is a loop), we find a curve from f^* to g^* that crosses e but no other edge (of G or G^*).

We call G^* the *planar dual* of the plane graph G.

→ If e is a bridge of G, then e is incident with just one face, so that e^* is a loop of G^*. Moreover, if e is a loop of G, then e^* is a bridge of G^*. That is, loops and bridges swap roles in the planar dual.

Recall that $d(f)$ denotes the degree of a face in a plane graph, where we count bridges twice.

Lemma 4.14 (The Handshaking Lemma for faces)**.** *Let G be a plane graph. Then*

$$\sum_{f \in F(G)} d(f) = 2|E(G)|.$$

Proof. For each face f of G, there is a vertex f^* in the dual G^*. Moreover, $d(f^*) = d(f)$. Therefore

$$\sum_{f \in F(G)} d(f) = \sum_{f^* \in V(G^*)} d(f^*).$$

There is a one-to-one correspondence between the edges of G and G^*. Using this fact, and the Handshaking Lemma, we obtain

$$\sum_{f \in F(G)} d(f) = \sum_{f^* \in V(G^*)} d(f^*) = 2|E(G^*)| = 2|E(G)|.$$

□

There is an obvious natural correspondence between the edges of G and the edges of G^*. Moreover, by construction, there is a natural correspondence between the faces of G and the vertices of G^*.

What is the relationship between the vertices of G and the faces of G^*? We might expect there to be a correspondence between these as well. However, there is a complication, as illustrated by the next exercise.

Exercise 4.15. *Find an example of a plane graph G with a planar dual G^* such that the number of vertices of G is not equal to the number of faces of G^*.*

The problem is, of course, due to lack of connectivity. For a graph G that solves Exercise 4.15, G is not connected. We will soon see that if we restrict our attention to connected graphs, all will be fine.

Before we do that, we note a curious fact. Observe that, for a graph G that solves Exercise 4.15, it is always the case that G^* is connected even though G is not connected.

Lemma 4.16. *If G is a plane graph, then G^* is connected.*

Proof. In this proof we use the fact that a path in G^* can be thought of as a sequence $f_1, f_2, \ldots, f_n$ of faces in G such that, for all $i \in \{1, 2, \ldots, n-1\}$, the faces f_i and f_{i+1} are adjacent.

We use induction on the number of *cycles* of G. If G has no cycles, then G is a forest. In this case, G has one face and G^* consists of loops attached to a single vertex. Thus G^* is connected.

Now assume that G has a cycle and that the result holds for graphs with fewer cycles than G. Let e be an edge in a cycle of G. As e is in a cycle, e is incident with two faces f_1 and f_2. In $G \backslash e$ these faces become a single face f. By the induction assumption, $(G \backslash e)^*$ is connected.

Assume that g and h are faces of G such that $\{g, h\} \cap \{f_1, f_2\} = \emptyset$. Since $(G \backslash e)^*$ is connected, there is a path P in $(G \backslash e)^*$ from g to h. If P does not contain f, then P is a path from g to h in G^*. Say P contains f. Let f' and f'' be the faces appearing immediately before and after f in P, respectively. In G, the faces f' and f'' are adjacent to either f_1 or f_2.

Up to symmetry, it suffices to consider the following two cases.

Case 1: f' and f'' are both adjacent to f_1. In this case, replace the subpath f', f, f'' of P by f', f_1, f'', to obtain a path from g to h in G^*.

Case 2: f' is adjacent to f_1 and f'' is adjacent to f_2. In this case we get the desired path by replacing the subpath f', f, f'' of P by f', f_1, f_2, f''.

We now consider the case where $\{g, h\} \cap \{f_1, f_2\} \neq \emptyset$. Observe that f_1, f_2 is a path from f_1 to f_2, so that handles the situation where $\{g, h\} = \{f_1, f_2\}$. We are left, up to symmetry, with the case where $g = f_1$ and $h \notin \{f_1, f_2\}$. In this case, let g' be a face other than f_2 that is adjacent to f_1. Then f_1, g' is a path. Moreover, by the earlier argument, there is a path in G^* from g' to h. Combining these paths gives the desired path from f_1 to h in G^*.

We have now shown that there is a path between every pair of vertices in G^* (or faces in G), so G^* is connected. The result follows by induction. $\square$

Connectivity Saves the Day: We get past any “Exercise 4.15 example”, by requiring that our plane graphs are connected.

For a vertex v of a graph G, let $\partial(v)$ denote the set of edges incident with v.

Lemma 4.17. *Let G be a connected plane graph.*

(i) *If v is a vertex of G, then $\{e^* : e \in \partial(v)\}$ is the set of edges in the boundary of a face of G^*.*
(ii) *If v^* is a face of G^*, then $\{e : e^* \in \partial(v^*)\}$ is the set of edges incident with a vertex in G.*

Via Lemma 4.17, we see that once we are in the world of connected graphs there is also a natural correspondence between the vertices of G and the faces of G^*.

Recall that for a plane graph G, we defined the dual G^* to have vertex set $V(G^*) = \{f^* : f \in F(G)\}$ and edge set $E(G^*) = \{e^* : e \in E(G)\}$.

We now consider G^{**}, by which we mean the graph $(G^*)^*$. We define $(x^*)^*$ to be x, for $x \in F(G) \cup E(G)$. In this way, the edge set of G^{**} is

$$E(G^{**}) = \{e : e^* \in E(G^*)\} = E(G),$$

and the vertex set of G^{**} is

$$V(G^{**}) = \{v : v^* \in F(G^*)\} = V(G).$$

Using this correspondence we now have:

Theorem 4.18. *If G is a connected plane graph, then $G^{**} = G$.*

We can say $G^{**} = G$ here so long as we use the natural correspondences between edges, vertices and faces discussed above. (We have not even defined isomorphism for plane graphs.)

Deletion-Contraction Duality. Contraction and deletion seem to be very different operations. But it turns out that they are related under duality.

Lemma 4.19. *Let G be a connected plane graph, and let e be an edge of G that is not a bridge. Then*

$$(G\backslash e)^* = G^*/e^*.$$

Proof. As e is not a bridge, it is incident to two distinct faces f_1 and f_2 in G. In $G\backslash e$ these two faces become a single face f. Moreover, a face g is adjacent to f in $G\backslash e$ if and only if it is adjacent to either f_1 or f_2 in G. Otherwise, the adjacencies of faces do not change.

In other words, in $(G\backslash e)^*$, the two vertices f_1^* and f_2^* merge to become a single vertex f^*. Moreover, a vertex g^* is adjacent to f^* if and only if it is adjacent to either f_1^* or f_2^* in G^*. Otherwise the adjacencies of vertices in G^* do not change. This is precisely the definition of the contraction of the edge e^* in G^*. $\square$

What is the lemma saying? It says that you get the same answer if you either

- start with G, delete e, and then take the dual; or
- start with G, take the dual, and then contract e^*.

We also have

Lemma 4.20. *Let G be a connected plane graph and let e be a non-loop edge of G. Then*
$$(G/e)^* = G^*\backslash e^*.$$

Proof. By Lemma 4.19, we have
$$(G^*\backslash e^*)^* = G^{**}/e^{**} = G/e.$$
Now
$$(G/e)^* = ((G^*\backslash e^*)^*)^* = G^*\backslash e^*.$$
□

We can apply Lemmas 4.19 and 4.20 to prove that 2-connectivity is preserved under duality, provided that the plane graph has enough faces. This turns out to be very useful.

Theorem 4.21. *Let G be a loopless 2-connected plane graph with at least three faces. Then G^* is loopless and 2-connected.*

We are almost in a position to prove Theorem 4.21. It will help to note one more lemma. The idea for this lemma is that while removing edges from a graph can reduce connectivity, putting edges back can only increase connectivity.

Lemma 4.22. *Let G be a graph with no bridges, and let e be an edge of G. If either $G\backslash e$ or G/e is 2-connected, then G is also 2-connected.*

Proof. [redacted] □

Proof of Theorem 4.21. Assume that $G = (V, E)$ is a 2-connected plane graph with no loops. Then G has at least three vertices and has no bridges, so it has at least three edges. If $|E| \leq 2$, then the theorem is easily checked to hold. Assume that $|E| > 2$ and, for induction, that the theorem holds for graphs with fewer edges than G.

As loops and bridges interchange under duality we know that G^* has no loops or bridges. Let e be an edge of G. By Theorem 3.4, either $G\backslash e$ or G/e is 2-connected. But if $G\backslash e$ is not 2-connected, then G/e is loopless, by Lemma 3.5. So either $G\backslash e$ or G/e is 2-connected and loopless.

Assume that $G\backslash e$ is 2-connected and loopless. By the induction assumption, $(G\backslash e)^*$ is 2-connected and loopless. By Lemma 4.19 $(G\backslash e)^* = G^*/e^*$. Thus G^*/e^* is 2-connected and loopless. By Lemma 4.22, G^* is 2-connected and loopless.

The argument is very similar when G/e is 2-connected and loopless. Assume that G/e is 2-connected and loopless. By the induction assumption, $(G/e)^*$ is 2-connected and loopless. By Lemma 4.19 $(G/e)^* = G^*\backslash e^*$. Thus $G^*\backslash e^*$ is 2-connected and loopless. By Lemma 4.22, G^* is 2-connected and loopless.

The result now follows by induction. □

Duality is Cool: Dualities of one type or another occur frequently in mathematics. Essentially, an operation Φ on a set X is said to be a "duality" if we always have $\Phi(\Phi(X)) = X$. Reflection in geometry is a duality. Taking inverses in algebraic structures — for example taking matrix inverses — give other dualitys. Having a duality is almost always useful.

One particular advantage of planar graph duality is that we can use it to go bargain hunting. In particular, we are often in a "buy one get one free" situation. We will give an example. First we need to learn a few things.

Recall that, for $X \subseteq V(G)$, we write $G[X]$ to denote the induced subgraph on vertex set X and edge set consisting of the edges of G whose ends are contained in X.

A *bond* of a connected graph is a minimal set of edges whose deletion disconnects the graph. For the remainder of this section, it is easier for us to view a cycle as a set of edges (rather than a subgraph). Then, it turns out that cycles and bonds interchange under duality. First we need a lemma.

Lemma 4.23. *Let G be a plane graph and let C be a cycle of G. Let X^* be the set of vertices of G^* that lie in* $\mathrm{int}(C)$*, and let Y^* be the set of vertices of G^* that lie in* $\mathrm{ext}(C)$*. Then $G^*[X^*]$ and $G^*[Y^*]$ are connected.*

For a subset S of $E(G)$, let S^* denote the subset $\{e^* : e \in S\}$.

Theorem 4.24. *Let G be a connected plane graph with planar dual G^*.*

(i) *If C is a cycle of G, then C^* is a bond of G^*.*
(ii) *If B is a bond of G, then B^* is a cycle of G^*.*

Proof. Consider (i). Say C is a cycle of $G = (V, E)$. Let X^* be the set of vertices of G^* in $\mathrm{int}(C)$ and let Y^* be the vertices in $\mathrm{ext}(C)$. We have $V^* = X^* \cup Y^*$. By Lemma 4.23, $G^*[X^*]$ and $G^*[Y^*]$ are connected graphs.

Every path in G^* from a vertex in X^* to one in Y^* must contain an edge in C^*, so $G^*\backslash C^*$ is not connected. So C^* is a set of edges that disconnects the graph G^*; next we will show it is a minimal set with the property. Let $e^* = u^*v^*$ be an edge of C^*, where $u^* \in X^*$ and $v^* \in Y^*$, and consider $G^*\backslash(C^* \setminus \{e^*\})$; in other words, we put an edge of C^* back into the graph. We have paths between pairs of vertices in X^* as $G^*[X^*]$ is connected and the same holds for pairs of vertices in Y^*. If x^* and y^* are vertices in X^* and Y^* respectively, then we get a path from x^* to y^* by combining a path from x^* to u^* with a path from v^* to y^* and using the edge e^*. This shows that $G^*\backslash(C^* \setminus \{e^*\})$ is connected for any $e^* \in C^*$, so C^* is a minimal set of edges that disconnects G^*. In other words, C^* is a bond of G^*, as required.

The proof of (ii) is left as an exercise. □

Now let's go bargain hunting. First of all we'll buy something.

Lemma 4.25. *Let G be a graph, and let C and D be edge sets of cycles of G that both contain the edge e. Then $(C \cup D) \setminus \{e\}$ contains the set of edges of a cycle.*

Proof. Say $e = uv$. Let $u, v, x_1, x_2, \ldots, x_s, u$ be the sequence of vertices traversed by the cycle with edge set C, and let $v, u, y_1, y_2, \ldots, y_t, v$ be the sequence of vertices traversed by the cycle with edge set D. Now consider a walk W along the following sequence of vertices: $u, y_1, \ldots, y_t, v, x_1, \ldots, x_s, u$. This is a closed walk as the vertex u is repeated, that traverses only edges in $(C \cup D) \setminus \{e\}$. If W is not a cycle, there is some other repeated vertex, so there is a proper subsequence of W, still traversing only edges in $(C \cup D) \setminus \{e\}$, that corresponds to a closed walk. Let W' be a minimal closed subsequence of W with this property. Then W' is a cycle, as otherwise contains a smaller closed walk. This shows that $(C \cup D) \setminus \{e\}$ contains the set of edges of a cycle. □

We now get the following for free.

Corollary 4.26. *Let G be a connected planar graph and let A and B be bonds of G that both contain the edge e. Then $(A \cup B) \setminus \{e\}$ contains a bond.*

Proof. Say A and B are bonds of G containing the edge e. By Theorem 4.24 A^* and B^* are cycles of G^* containing the edge e^*. By Lemma 4.25, there is a cycle C^* of G^* contained in $(A^* \cup B^*) - \{e^*\}$. By Theorem 4.24, $(C^*)^* = C$ is a bond of G contained in $(A \cup B) \setminus \{e\}$. □

Wow! That was a real bargain. And there are plenty of others to be hunted.

→ You might notice that Corollary 4.26 holds even if G is not planar. Tutte observed the same thing back in the 1940s. In fact, he noticed that a lot of results that you get for free for planar graphs, using planar graph duality, also worked for non-planar graphs. He decided that there must be a more general duality going on that works even when graphs are non-planar. This led to his interest in the theory of matroids. Those of you who plan to do honours can find out all about matroids by enrolling in MATH432.

Who was Bill Tutte? One of the greatest mathematicians of the twentieth century, Tutte proved many deep theorems in graph theory as well as fundamental theorems in matroid theory. He is the unsung hero of the Bletchley Park codebreakers in the Second World War as you can see **here**. His **Wikipedia page** makes interesting reading.