Contraction Degeneracy

Gordon Royle

Centre for the Mathematics of Symmetry and Computation
School of Mathematics & Statistics
University of Western Australia

Geoff-Fest 2015

Meeting Geoff 2004

Acknowledgements

This talk describes *work in progress*, initially stimulated by the paper

- *Fijavž & Wood, Graph Minors and Minimum Degree*.

and which has benefitted from discussions with

- *Irene Pivotto*,
- *Douglas Woodall*,
- Insert $\langle$*Your name*$\rangle$ here!

DEGENERACY

The *degeneracy* $D(G)$ of a graph G is defined by

$$D(G) = \max_{H \subseteq G} \delta(H),$$

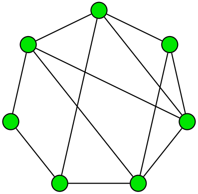
where $\delta(H)$ is the *minimum degree* of H and the maximum is taken over all *subgraphs* of G.

Greedy Algorithm

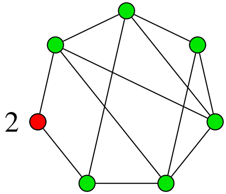
Although the maximum is over all subgraphs, it is well known and easy to show that a *greedy algorithm* can be used to determine $D(G)$:

- Repeatedly delete a vertex of *minimum degree*
- Record the degree of each vertex as it is deleted
- Take the *maximum* of the recorded values

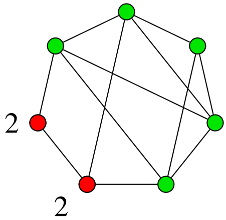
EXAMPLE



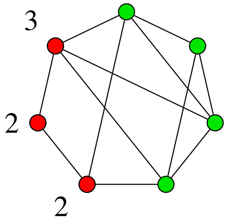
EXAMPLE



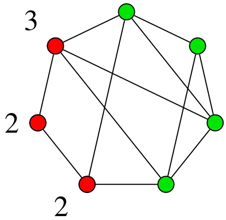
Example



EXAMPLE



Example



And we can stop now and conclude that $D(G) = 3$.

Chromatic Number

It is also well known that

$$\chi(G) \leq D(G) + 1.$$

A $D(G) + 1$-colouring of the graph can be determined just by remembering the order in which vertices were removed to determine $D(G)$, and then colouring them in *reverse order*.

The value $D(G) + 1$ is often called the *colouring number* of G.

Bounded Degeneracy

Degeneracy is one of parameters used by *fixed-parameter complexity theorists* to stratify graphs into classes of increasing "complexity" for algorithmic purposes.

- *Forests* are 1-degenerate, and conversely
- *Series-parallel* graphs are 2-degenerate (but not conversely)
- *Planar* graphs are at most 5-degenerate (but not conversely)

Matula number

In 1968, *David Matula* defined the parameter

$$D_m(G) = \max_{H \subseteq G} \kappa'(H),$$

where $\kappa'(H)$ is the size of the minimum *edge-cutset* of H.

- $D_m(G)$ can be calculated by the greedy algorithm
- A graph can be coloured in $D_m(G) + 1$ colours
- $D(G)$ can be arbitrarily greater than $D_m(G)$ (Woodall)

For matroids?

If M is a matroid, then we can define

$$D_m(M) = \max_{N \in R(M)} \min_{C^* \in \mathcal{C}^*(N)} |C^*|$$

where

- $R(M)$ is the set of *restriction minors* of M
- $\mathcal{C}^*(N)$ is the set of cocircuits of N

James showed that the chromatic number of a *regular* matroid is bounded by $D_m(M) + 1$.

Contraction Degeneracy

The *contraction degeneracy* $C(G)$ of a graph G was explicitly named by *Bodlaender, Koster & Wolle*:

$$C(G) = \max_{H \preceq G} \delta(H),$$

where $\delta(H)$ is the *minimum degree* of H and the maximum is taken over all *simple minors* of G.

Unlike degeneracy, $C(G)$ is NP-hard to compute.

Fijavž & Wood have recently studied the families of graphs

$$\mathcal{X}_k = \{G \mid C(G) \leq k\}.$$

Each $\mathcal{X}_k$ is *minor-closed* and so enjoys[1] all of the properties implied by the *Graph Minor Theory* of *Robertson & Seymour*.

So $\mathcal{X}_k$ can be defined by a *finite* collection of *excluded minors*:

$$\widehat{\mathcal{X}}_k = \{G \mid C(G) > k \text{ and } C(H) \leq k \text{ for all } H \prec G\}$$

[1] or should this be "endures"?

Some basics

- The class $\mathcal{X}_1$ consists of all *forests* and so $\widehat{\mathcal{X}}_1 = \{K_3\}$.
- The class $\mathcal{X}_2$ is the *series-parallel graphs* and so $\widehat{\mathcal{X}}_2 = \{K_4\}$.
- The class $\mathcal{X}_3$ has

  $$\widehat{\mathcal{X}}_3 = \{K_5, K_{2,2,2}\}$$

  where $K_{2,2,2}$ is the *octahedron*.
- The class $\mathcal{X}_4$ has an unknown collection of excluded minors, but *Fijavž & Wood* list nine in their paper.

Problem

Determine $\widehat{\mathcal{X}}_4$

A graph G on n vertices is *k-realizable* if for every embedding

$$\varphi : G \to \mathbb{R}^n$$

there is an embedding

$$\varphi' : G \to \mathbb{R}^k$$

with the same edge lengths.

Maria Belk and Robert Connelly have shown that for $k = 1, 2, 3$ the class $\mathcal{X}_k$ is equal to the class of *k-realizable* graphs.

The *chromatic polynomial* $P(G,q)$ of a graph G counts the number of proper q-colourings of G for integer values of q.

- (Woodall 1997, Thomassen 1997)
  If every simple minor of a graph G has a vertex of degree at most d, then $P(G,q) > 0$ for all *real* $q \in (d, \infty)$.
- (Mader 1967), There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph with minimum degree greater than $f(d)$ has K_d as a minor.

Therefore any *proper minor-closed* family of graphs has bounded *real* chromatic roots.

For matroids, we can define a parameter analogous to the contraction degeneracy just by replacing *minimum degree* with *minimum cocircuit size* and *chromatic polynomial* with *characteristic polynomial*:

We also have:

- (Oxley 1978) [2] If every simple minor of a matroid M has a cocircuit of size at most d, then $P(M, q) > 0$ for all real $q \in (d, \infty)$.

[2] With a prescience impressive even for James, he proved this more general result two decades earlier than Woodall and Thomassen.

MOST WANTED ANSWER

QUESTION

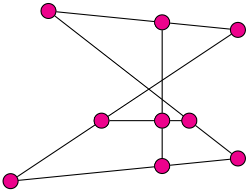
Is it true that any *minor-closed class* of binary matroids, not including all graphs, has bounded *real* chromatic roots?

In particular, take the class to be *cographic* matroids, and resolve Welsh's question about whether *real flow roots* of graph are bounded.

This seems *very plausible*, but any proof would need to be quite different to that for graphs as there is no analogue of Mader's result.

Let $\mathcal{M}_k$ denote the set of *binary matroids* with the property that every *simple minor* has a *cocircuit* of size at most k.

- $\mathcal{M}_1$ is the class of all *forests*
- $\mathcal{M}_2$ is the class of *series-parallel graphs*
- $\widehat{\mathcal{M}}_3 = \{K_5, K_{2,2,2}, F_7, K_{3,3}^*\}$[3]

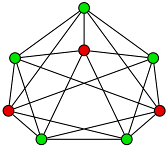


[3] I am no longer sure where I learned this

Back to $\widehat{\mathcal{X}}_4$

Some excluded minors for $\widehat{\mathcal{X}}_4$ can immediately be found:

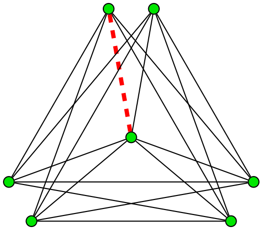
- The complete graph K_6
- The complete multipartite graph $K_{2,2,2,1}$
- The complete join $C_5 \vee 3K_1$



- The icosahedron I

THEOREM (MADER)

A graph with minimum degree 5 has one of K_6, $C_5 \vee 3K_1$, I or $K_{2,2,2,1} - e$ as a minor.

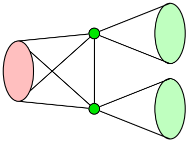


So the remaining excluded minors for $\mathcal{X}_4$ must have $K_{2,2,2,1} - e$ as a minor, but not $K_{2,2,2,1}$.

More properties

Suppose G is an excluded minor for $\mathcal{X}_4$. Then

- G has minimum degree 5,
- the vertices of *high degree* form an *independent set*[4],
- every edge of G lies in a *triangle* with a *low degree* vertex.



The *common neighbours* of the endpoints of the edge are the only vertices whose degree *drops* on contraction.

[4] possibly empty

The number of graphs of minimum degree 5 grows very rapidly. However Brendan McKay's `geng` program, which generates graphs vertex-by-vertex, allows a user-specified *pruning function* to be defined.

So the search is pruned as soon as there are:

- Adjacent vertices, each of degree more than 5
- Adjacent vertices, each of degree at least 5, with no common neighbour

The survivors are checked for simple *necessary conditions*, with an *exhaustive program* used for the final test.

Up to 14 vertices

The search is *complete* up to 14 vertices. Let $e(n)$ denote the number of edges of the smallest graph on n vertices with minimum degree 5.

n	$e(n)$	$+0$	$+1$	$+2$	$+3$
6	15	1			
7	18	1			
8	20	1			
9	23				
10	25		1		
11	28	1	1		
12	30	3	4	4	
13	33	9	16	6	
14	35	6	31	18	4

Is this *escalating out of control* or settling down?

By restricting to *nearly* 5-regular graphs, we can go a bit further

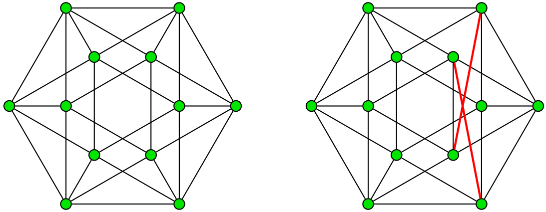
n	$e(n)$	$+0$	$+1$	$+2$	$+3$
6	15	1			
7	18	1			
8	20	1			
9	23				
10	25		1		
11	28	1	1		
12	30	3	4	4	
13	33	9	16	6	
14	35	6	31	18	4
15	38	27	31	9	1

This is not looking promising

PERHAPS OK?

But if we look at the *connectivity* of the graphs, some patterns emerge.

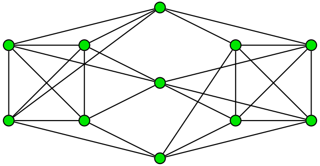
Exactly six of the graphs are 5-connected: K_6, $K_{2,2,2,1}$, $C_5 \vee 3K_1$, an ugly 10-vertex graph, the icosahedron and a mutant icosahedron.



The rest

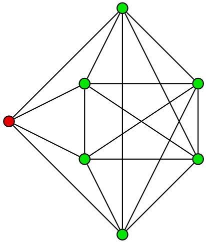
The remaining examples all have

- Connectivity exactly 1, 2 or 3
- A *unique* set of vertices of size κ whose removal disconnects the graph



Gluing

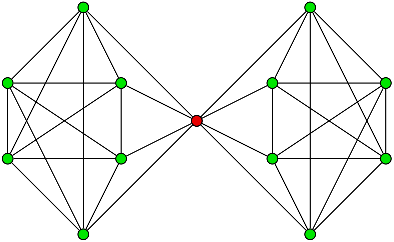
All of the excluded minors arise from gluing together *gadgets* in any way that avoids the obvious forbidden configurations.



A gadget has minimum degree 5 except for a distinguished set L of low-degree vertices, such that every contraction minor has a vertex of degree at most 4 outside L.

GLUING

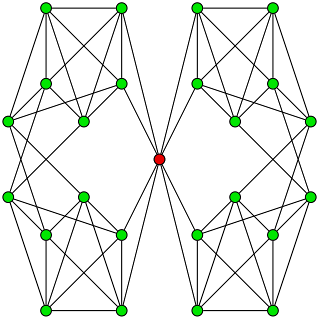
All of the excluded minors arise from gluing together *gadgets* in any way that avoids the obvious forbidden configurations.



A gadget has minimum degree 5 except for a distinguished set L of low-degree vertices, such that every contraction minor has a vertex of degree at most 4 outside L.

Can we finish?

I believe that the largest excluded minor has 25 vertices.



If I knew this for sure, I think that finding the remaining excluded minors could be done.

What about matroids?

I know some interesting binary matroids in $\widehat{\mathcal{X}}_4$, such as $M^*(P)$.

The icosahedron shows that we're looking at matroids of rank up to 11 and size 30 at least, which is unlikely to work.

Happy Birthday

Thanks for listening and Happy Birthday Geoff!