3-FLOWS WITH LARGE SUPPORT

A conference in honour of Geoff Whittle

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Joint work with M. DeVos, J. McDonald, E. Rollova, R. Šamal

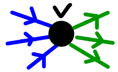
Flows

G : a graph with an orientation.

Γ : an additive abelian group

A **Γ-flow** is a function $\phi : E(G) \longrightarrow \Gamma$ s.t.

$$\forall\, v \in V(G), \quad \sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e)$$

$\delta^-(v)$  $\delta^+(v)$

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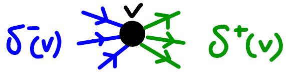
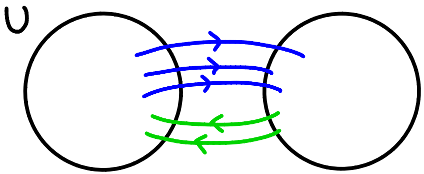


This implies $\displaystyle\sum_{e \in \delta^+(U)} \phi(e) = \sum_{e \in \delta^-(U)} \phi(e) \quad \forall\ U \subseteq V(G)$



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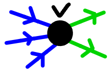
$\delta^+(v)$  $\delta^-(v)$

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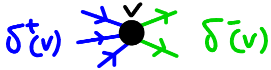

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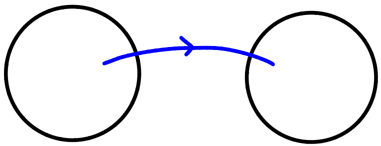
If G has a bridge then G has no nowhere zero flow :



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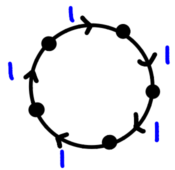
$h(G,K) = 1$ iff G has a nowhere zero flow

$h(G,K) = 0$ iff G is a forest



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What we know:

- $h(2j,2) = h(2j+1,2) = \dfrac{2j}{2j+1}$
- $h(3,4) = \frac{14}{15}$ (Kral, Kaiser and Norine)
- $h(4,4) = 1$ (Jaeger)
- $h(2,6) = 1$ (Seymour)
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What we'd like to know:

- Tutte 3-flow conj: $h(4,3) = 1$
- Tutte 5-flow conj: $h(2,5) = 1$
- Other values?

$h(t,k)$ is non-decreasing in both t and k.

$t \backslash k$	2	3	4	5	6
2	$2/3$	$\geq \frac{3}{4}$ (Kral)		Tutte 5-flow conj.: 1	1 (Seymour)
3	$2/3$		$14/15$ (KKN)		1
4	$4/5$	Tutte 3-flow conj.: 1	1 (Jaeger)	1	1
5	$4/5$		1	1	1
6	$6/7$	1 (LTWZ)	1	1	1

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Theorem: every 3-edge-connected graph G has a 3-flow ϕ with

$$|\mathrm{supp}(\phi)| \geq \frac{5}{6} |E(G)|$$

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Corollary: if G is a simple planar graph, then there exists a colouring $f : V(G) \to \{1,2,3\}$ so that the number of edges uv with $f(u) = f(v)$ is at most $\frac{1}{6} |E(G)|$.

Given $\phi : E(G) \longrightarrow \Gamma$, define $\partial\phi(v) = \sum_{e \in \delta^-(v)} \phi(e) - \sum_{e \in \delta^+(v)} \phi(e)$.

If ϕ is a flow, $\partial\phi(v) = 0 \ \forall \ v \in V(G)$.

In general, $\sum_{v \in V(G)} \partial\phi(v) = 0$.

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Theorem: if G is an oriented 3-edge-connected graph and $\mu : V(G) \to \mathbb{Z}_3$ is zero-sum, then $\exists\ \phi : E(G) \to \mathbb{Z}_3$ s.t.:

- $\partial\phi = \mu$, and
- $|\mathrm{supp}(\phi)| \geq \frac{5}{6} |E(G)|$.

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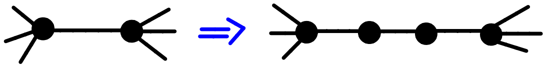
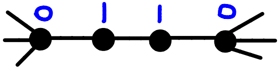
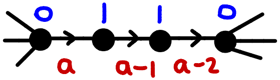
Here 3-edge-connectivity is necessary:

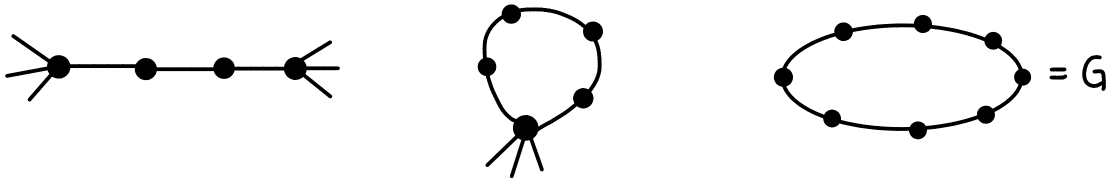
- take a 3-edge-connected graph G with $|E(G)| = 0 \pmod 3$
- Subdivide each edge of G twice: 
- Set μ as 

Any ϕ will look as 

so $|\mathrm{supp}(\phi)| \leq \frac{2}{3} |E(G)|$

We move to subdivisions of 3-edge-connected graphs.

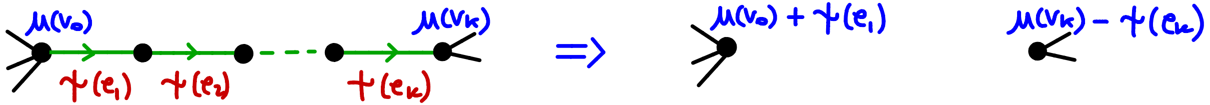
An ear in a 2-edge-connected graph is a subgraph that is one of:



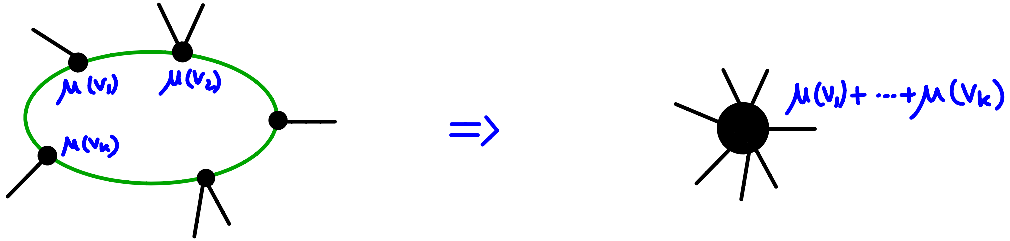
We can partition the edge set of G into edge sets of ears.

Possible reductions :

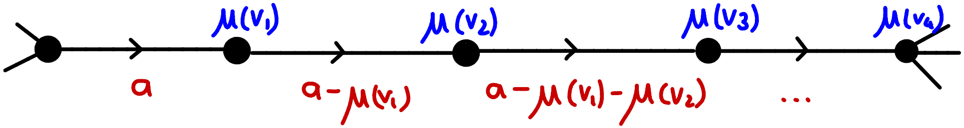
- deleting ears



- contracting 2-edge-connected subgraphs



For a valid ϕ, once we choose the value of ϕ on one edge of an ear P, all the values on that ear are fixed.



Different types of ears help us in different ways (or don't help at all!)

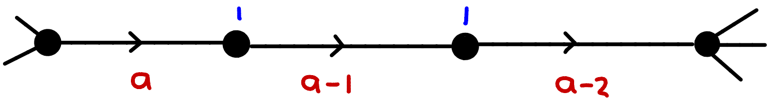
So we assign different bonuses to different ears.

The bonus of an ear P is :

$$\text{bonus}(P) = \begin{cases} 0 & \text{if } P \text{ equitable} \\ 3 & \text{if } |E(P)| \equiv 2 \pmod 3 \\ 4 & \text{otherwise} \end{cases}$$

Equitable ear P: for any choice of ϕ, $\frac{1}{3}$ of the edges of P get value zero.

e.g.



bonus (G) = Sum of the bonuses of all the ears.

If H is a subgraph of G and $\psi : E(H) \longrightarrow \mathbb{Z}_3$ then

$$\text{gain}(\psi) = 24\,|\text{supp}(\psi)| - 16\,|E(H)|$$

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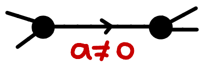
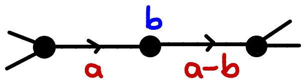
Le: if P is an inequitable ear of G and $|E(P)| = 3k+i$ $(1 \leq i \leq 3)$, then P has a valid $\psi : E(P) \to \mathbb{Z}_3$ with $\text{gain}(\psi) \geq 8i$.

e.g.



gain $= 24 - 16 = 8$



gain $= 2\cdot 24 - 2\cdot 16 = 16$

$a,\ a-b \neq 0$

Lemma: if G is an oriented graph that is a subdivision of a 3-edge-connected graph and $\mu : V(G) \longrightarrow \mathbb{Z}_3$ is zero-sum, then $\exists\ \phi : E(G) \longrightarrow \mathbb{Z}_3$ s.t.:

- $\partial\phi = \mu$, and
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Lemma: if G is an oriented graph that is a subdivision of a 3-edge-connected graph and $\mu: V(G) \to \mathbb{Z}_3$ is zero-sum, then $\exists\ \phi: E(G) \to \mathbb{Z}_3$ s.t.:

- $\partial\phi = \mu$, and
- $\text{gain}(\phi) \geq \text{bonus}(G)$.

If G is 3-edge connected, then every edge of G is an ear of length 1 and $\text{bonus}(G) = 4|E(G)|$.

So $\text{gain}(\phi) = 24|\text{supp}(\phi)| - 16|E(G)| \geq 4|E(G)|$

$$\Rightarrow |\text{supp}(\phi)| \geq \frac{5}{6}|E(G)|.$$

Thank you!!