The Quest for Rota's Conjecture

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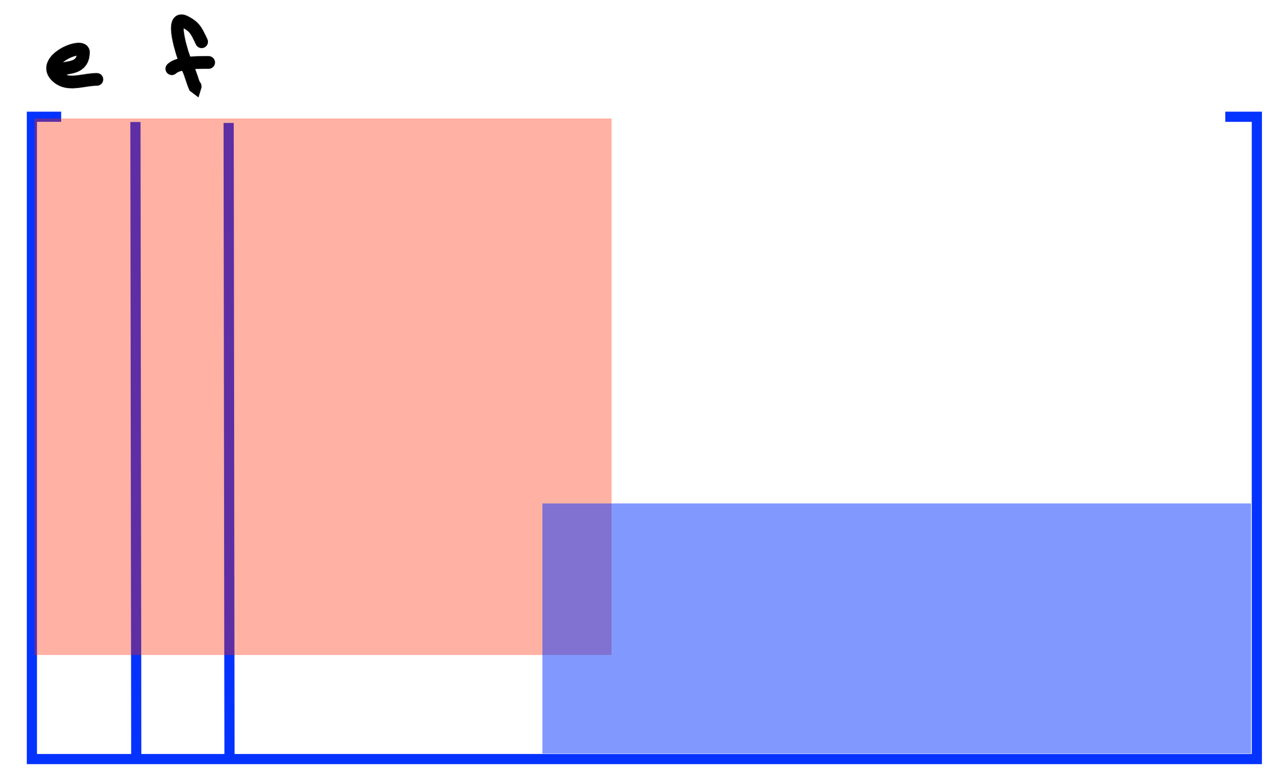
Rota's Conjecture [1970]

For each finite field $\mathbb{F}$, there are only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.

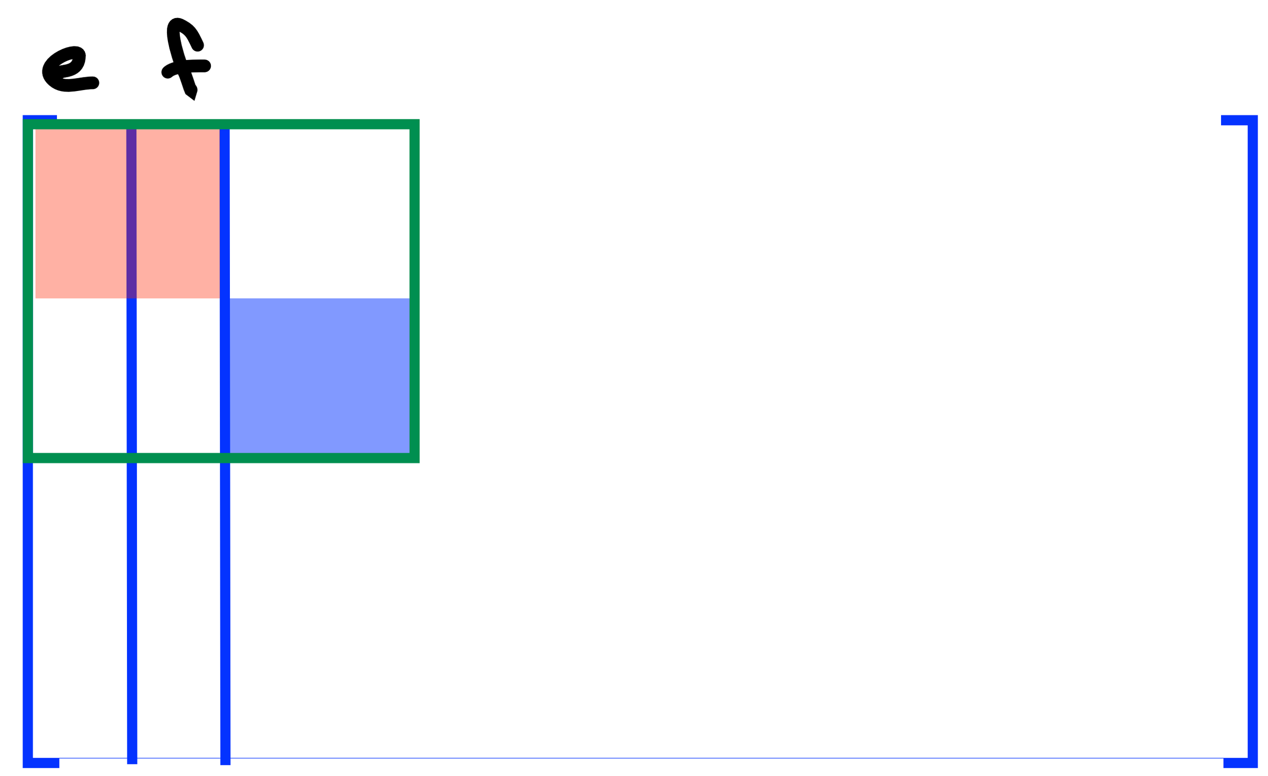
Proof for GF(4)

(1) Suppose that $M \backslash e, f$ is 3-connected with a $U_{2,4}$-minor

(2) Construct a GF(4)-representable matroid $\tilde{M}$ with $\tilde{M} \backslash e = M \backslash e$ and $\tilde{M} \backslash f = M \backslash f$.



(3) Find a small minor with both the "bad determinant" and and the "stabilizer"
(intertwining)

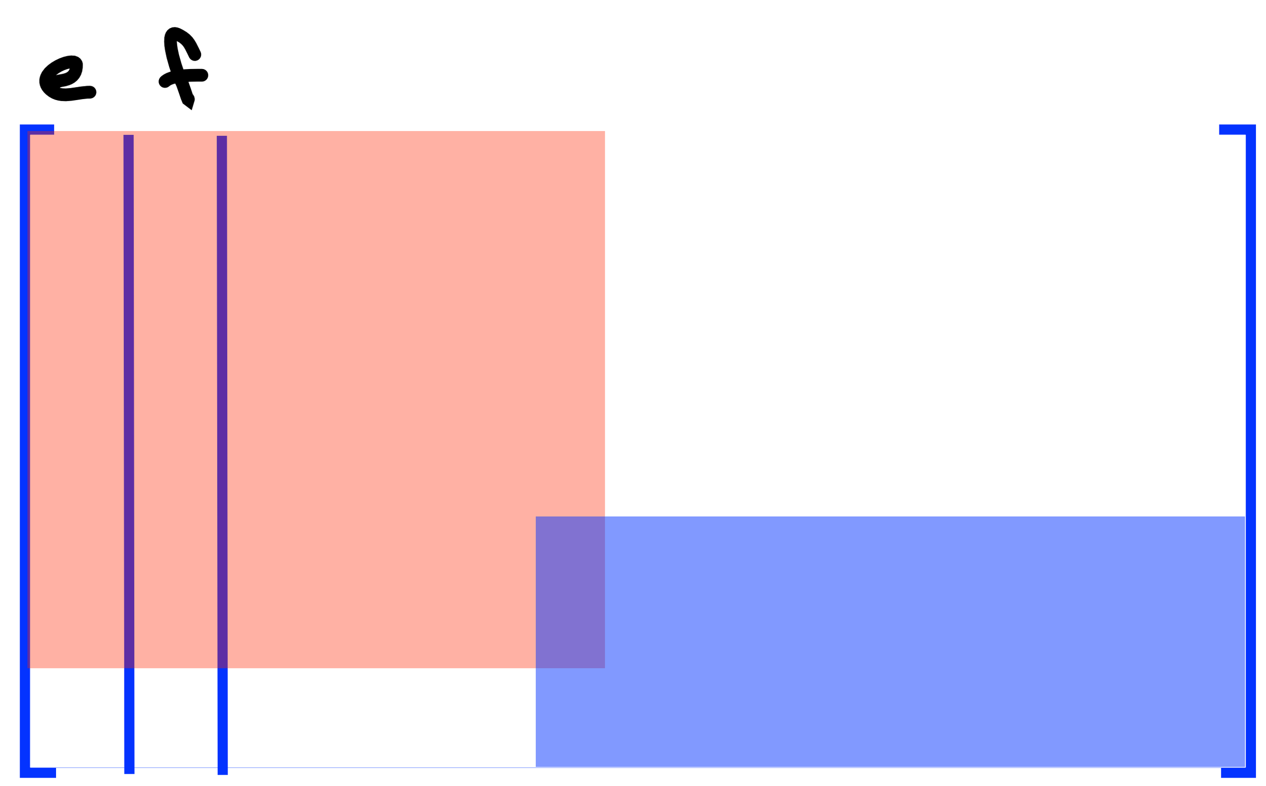


(4) Patch the 2-separations in the minor, (connectivity)

Obstacles for GF(5)

(1) six inequivalent representations

(2) intertwining



Bigger obstacles to come

(1) [OVW 1996]

For $|\mathbb{F}| \geq 7$, there exist 3-connected matroids with arbitrarily many inequivalent representations over $\mathbb{F}$.

(2) [Vertigan, unpublished]

For a "typical" pair (N_1, N_2) of matroids, there are infinitely many matroids that are minor-minimal subject to containing both N_1 and N_2 as a minor.

Problem [1997]

For $\mathbb{F} \in \{GF(7), GF(8), GF(9), \ldots\}$ prove that 4-connected matroids have at a bounded number of inequivalent representations over $\mathbb{F}$.

Theorem [GW]

If M is a sequentially 4-connected matroid with $|M| \geq 4$, and M is not a wheel or a whirl than then there exists $e \in E(M)$ such that $M \backslash e$ or M/e is sequentially 4-connected.

Oberwolfach [1999]

Conjecture [J R S]

For $k \gg n$, if M has branch-width $\geq k$, then either

(i) M has a $U_{n,2n}$-minor,

(ii) M has an $M(n \times n\text{-grid})$-minor, or

(iii) M or M^* has a $BM(n \times n\text{-grid})$-minor.

Fields Institute [1999]

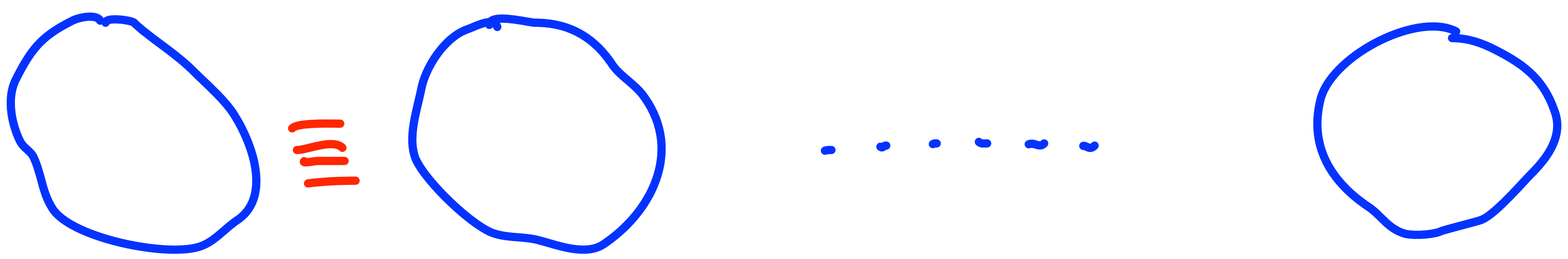
Theorem [GGW]

For each $k \geq 0$, the set of $\mathbb{F}$-representable matroids with branch-width $\leq k$ is well-quasi-ordered.

Toward a grid theorem

Problem [2000]

If M is an $\mathbb{F}$-representable matroid with huge branch-width, does M or M^* have many skew circuits pairwise highly connected?



C_1 C_2 C_n

Hard-fought progress

Theorem [GGW 2001]

If M is an $\mathbb{F}$-representable matroid with sufficienly large rank, then either

(a) M has k disjoint cocircuits, or

(b) M has an $M(K_n)$-minor.

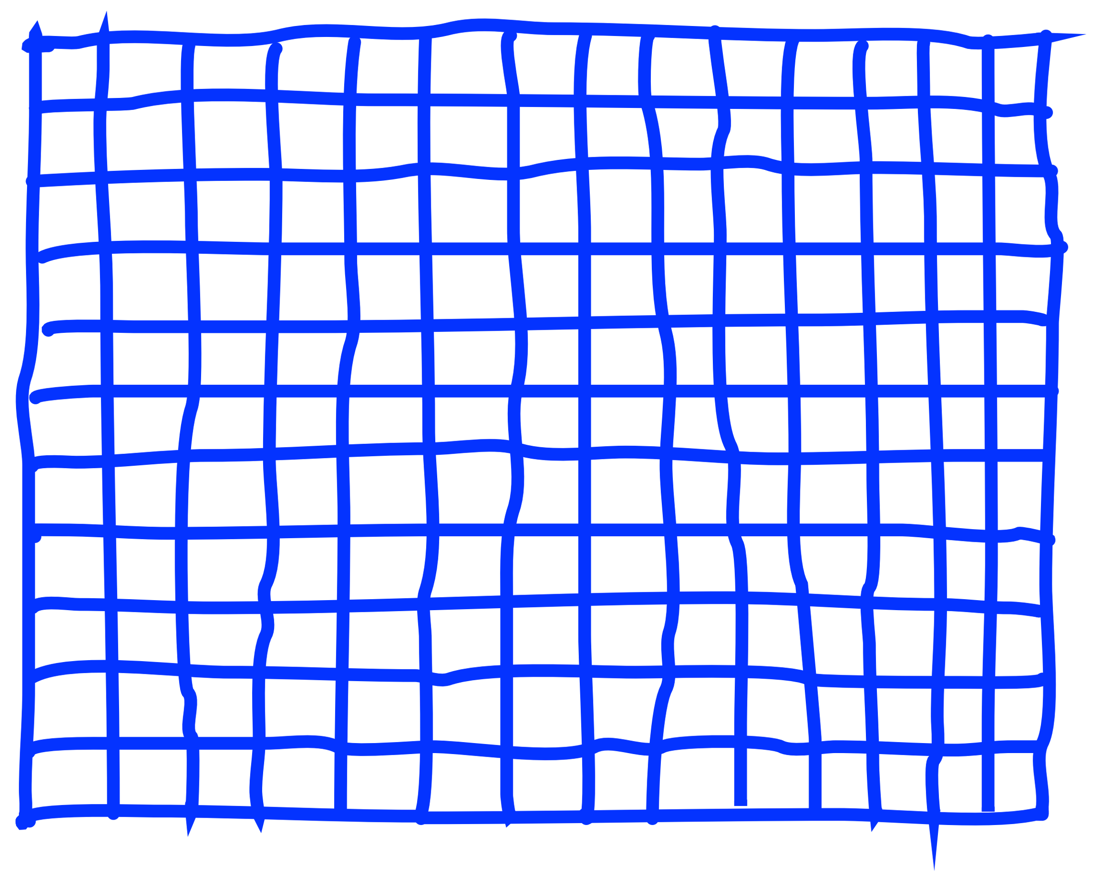
$\Rightarrow$ Erdős - Posá Theorem

Corollary [GW 2002]

If M is a simple $\mathbb{F}$-representable with no $M(K_n)$-minor, then $|M| \leq c(n, \mathbb{F}) \cdot r(M)$.

At last, a grid theorem!

Theorem [GGW 2002]
If M is an $\mathbb{F}$-representable matroid with sufficiently large branch-width, then M contains the cycle matroid of an $n \times n$-grid as a minor.



Problem [2003]

Can we solve Rota's Conjecture now that we have a grid theorem? GF(5)? Dyadic?

Back to inequivalent representations

Theorem [GW 2008]

If $|\mathbb{F}|$ is prime, then every 4-connected matroid has a bounded number of inequivalent representations over $\mathbb{F}$.

Theorem [GGW 2008]

For each k there is a finite field $\mathbb{F}$ such that there are k-connected matroids with arbitrarily many inequivalent representations over $\mathbb{F}$.

Matroid Minors Structure Theory [2003 - 2010]

Let $p = \operatorname{char}(\mathbb{F})$ and $n_0 \gg n_1 \gg n_2 \gg n_3$.

(I) M has an $M(n_0 \times n_0\text{-grid})$
but no $M(K_{n_1})$- nor $M(K_{n_1})^*$-minor

(II) M has an $M(K_{n_1})$-minor
but no $PG(n_2-1, p)$-minor

(III) M has a $PG(n_2-1, p)$-minor
but no $PG(n_3-1, \mathbb{F})$-minor

Applications

(1) WQO Theorem

There is no infinite antichain of $\mathbb{F}$-representable matroids.

(2) Fragility Theorem

If M is an $\mathbb{F}$-representable N-fragile matroid then branch-width$(M) \leq c(N, \mathbb{F})$.

Problem [2011]

Can we solve Rota's Conjecture now?
When $|\mathbb{F}|$ is prime?

GF(5) (Yes, Van Zwam + Intertwining)

Theorem [GGW 2011]

If (A,B) is a k-separation in an excluded minor, then $\min(|A|,|B|) \le c(k,\mathbb{F})$.

Graded Connectivity

Let $\ell = (\ell_1, \dots, \ell_{k-1}) \in \mathbb{Z}^{k-1}$.

M is ℓ-connected if for each t-separation (A,B), with $t<k$, $\min(|A|,|B|) \le \ell_t$.

Theorem [GGHZ 2012]

Let $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^{k-1}$ with $0 \le \ell_1 \ll \ell_2 \ll \cdots \ll \ell_{k-1}$.
If M is a sufficiently large ℓ-connected matroid then M has an ℓ-connected minor N with $|N| \in \{|M|-1, |M|-2\}$.

Corollary [GGHZ 2012]

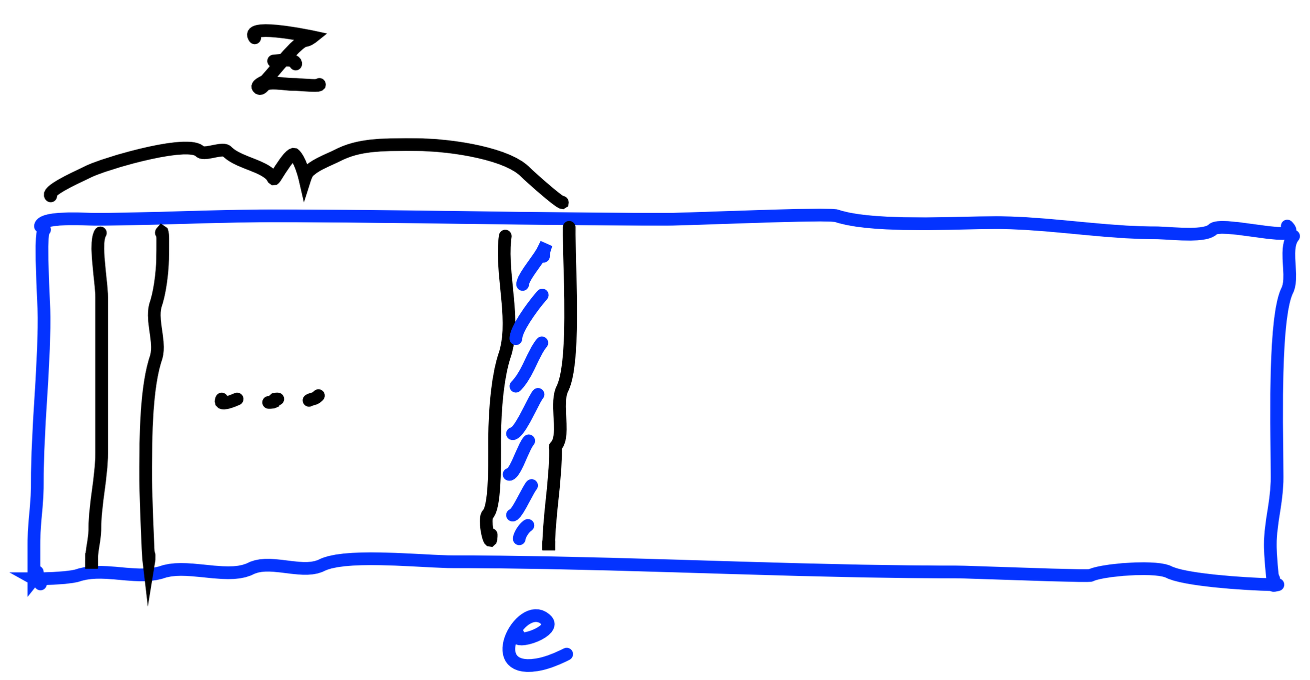
For each $\mathbb{F}$ there exists k such that k-connected matroids have a bounded number of inequivalent representations over $\mathbb{F}$.

Highly connected excluded minors

Let M be an excluded minor.

Suppose that M is k-connected where $k \gg |\mathbb{F}|$ and that $|M| \gg |\mathbb{F}|$.

$\Rightarrow$ There is a large set $Z \subseteq E(M)$ and a representation of $M \setminus Z$ that extends to a representation of $M \setminus x$ for each $x \in Z$

$A =$ 

$\Rightarrow$ $M(A) \setminus x = M \setminus x$ for each $x \in Z$.

$\Rightarrow$ $(M(A)/e) \setminus x = (M/e) \setminus x$ for each $x \in Z$

Note: $M(A)/e$ & M/e are both $\mathbb{F}$-representable.

The final step

Theorem [GGW 2013]

If M_1 and M_2 are $\mathbb{F}$-representable matroids on a common ground set E and M_1 and M_2 differ in rank on a single subset of E, then branch-width$(M_1) \leq c(|\mathbb{F}|)$.

Open Problems

(1) For $\mathbb{F} \in \{GF(4), GF(5), \dots\}$, describe the $\mathbb{F}$-representable matroids with a circuit-hyperplane whose relaxation leaves an $\mathbb{F}$-representable matroid.

(2) For prime $\mathbb{F}$, find an easier proof of Rota's Conjecture (using the Fragility Theorem but no other Matroid Minors machinery)

(3) Find a nice algebraic proof for the "Intertwining Theorem" for $\mathbb{F}$-representable matroids.

Happy Birthday Geoff!