Beyond matroids?
Permutation group bases and Boolean representable complexes

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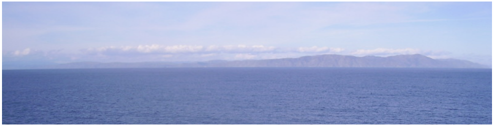

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Happy birthday, Geoff!

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Thus, the symmetric group S_n can be generated by two elements, but it also has a minimal (with respect to inclusion) generating set of size $n-1$; and indeed, a theorem of Julius Whiston shows that this is best possible.

But Philippe Cara and I showed that minimal generating sets of size $n-1$ for S_n are derived from labelled trees on n vertices in a simple way; these objects are not so badly behaved ...

Set families

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Let $\mathcal{F} = (F_e : e \in E)$ be a family of subsets of a set X. We use intersection to define a matroid-like structure on E. Thus, we say that an irredundant base is a sequence $(e_1, \ldots, e_k)$ with the properties:

- $\displaystyle\bigcap_{j=1}^{k} F_{e_j} = \bigcap_{e \in E} F_e$;
- for $j = 1, \ldots, k$, F_{e_j} does not contain $\displaystyle\bigcap_{l=1}^{j-1} F_{e_l}$.

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The first condition defines the notion of spanning, the second the notion of independent.

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In particular, every matroid can be represented by an IBIS family of sets.

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A base is irredundant if no point is fixed by the pointwise stabiliser of its predecessors.
Bases are used in computational group theory: every element of G is uniquely determined by its effect on a base, so a small base gives an efficient representation of group elements.

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The IBIS groups form a generalisation of the automorphism groups of independence algebras.

Open questions

- Which matroids arise from IBIS families of subgroups of a group? (If the family of subgroups is closed under conjugation, then the group can be represented as an IBIS permutation group; not all matroids can occur here.)

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- Which matroids arise from IBIS families of subgroups of a group? (If the family of subgroups is closed under conjugation, then the group can be represented as an IBIS permutation group; not all matroids can occur here.)
- What about IBIS families of substructures of other kinds of structure (subrings of a ring, sub-Latin squares of a Latin square, etc.)?
- Which set families, or permutation groups, have the weaker property that all minimal bases have the same size? The minimal bases need not be the bases of a matroid in this case; is there a theory of the structures that arise in this way?

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The notion of Boolean representable simplicial complexes developed in the book includes all matroids, but not all simplicial complexes; and it is the authors' claim that these objects have a rich structure theory. They invite readers

> to consider the new class of boolean representable simplicial complexes as a brave new world to explore (we believe many of the theorems in matroid theory will extend to boolean representable simplicial complexes).

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I will spend the rest of the time on a very brief introduction to Boolean representable complexes. But don't take this as gospel; go and read the book instead!

The boolean and superboolean semirings

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Now the boolean semiring $\mathbb{B}$ is $\mathbb{N}_1$, and the superboolean semiring $\mathbb{SB}$ is $\mathbb{N}_2$.
Note that $\mathbb{B}$ is not a subsemiring of $\mathbb{SB}$; it is a quotient of it.

The addition and multiplication tables of $\mathbb{B}$ are

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
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The tables for $\mathbb{SB}$ are

$+$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$\cdot$	0	1	2
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1	0	1	2
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A superboolean representation of a simplicial complex is a matrix whose independent sets of columns are the simplices. A boolean representation is one given by a matrix with entries 0 and 1 only. Note that even in the boolean case, independence is defined over $\mathbb{SB}$.

A small example

Let G be the cyclic group of order 6 generated by

$$(1,4)(2,5,6)(3,7,8,9,10,11).$$

The irredundant bases are, up to parallel elements, $(1,2)$ and (3). So the "independent sets" are the empty set, the singletons, and $\{1,2\}$.

Define a matrix with rows and columns indexed by $1,2,3$, with (i,j) entry 1 if $G_i = G_j$, 2 if $G_i < G_j$ or *vice versa*, and 0 if G_i and G_j are incomparable. The matrix is

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Warning: This is a superboolean, not a boolean representation. Also, this simple construction doesn't work for all permutation group bases!

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