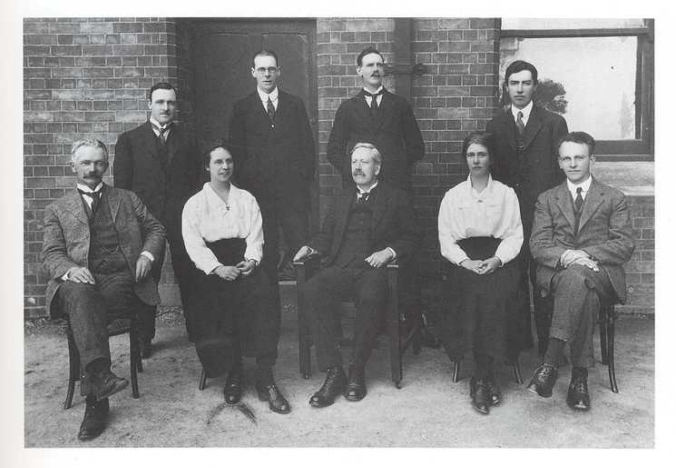
Mereocompactness

Rob Goldblatt

Duncan Sommerville

Professor of Pure and Applied Mathematics, Victoria College, 1915–1934



Science Faculty 1919

Mereotopology:

abstract study of qualitative relations between regions of space.

- x is connected to y: xCy
- x is a part of y:

$$xPy \quad \text{iff} \quad \forall z(zCx \to zCy)$$

- x overlaps y:

$$xOy \quad \text{iff} \quad \exists z(zPx \;\&\; zPy)$$

- x is externally connected to y:

$$xECy \quad \text{iff} \quad xCy \;\&\; \text{not } xOy$$

- x is a non-tangential part of y:

$$xPy \text{ and not } \exists z(zECx \;\&\; zECy)$$

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Whitehead 1919/20

extends over relation between events
(converse to part of relation)

de Laguna 1922

can connect relation between solids

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extensive connection relation between regions

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Formalised Whitehead's theory as a first-order calculus with the primitive relation is connected to

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Computer Scientists

Randall and Cohn 1989, Randall, Cui and Cohn 1992

initiated the development of Clarke's calculus into a Regional Connection Calculus (RCC) for qualitative spatial reasoning, incorporating aspects of Leśniewski's mereology.

Gotts 1996

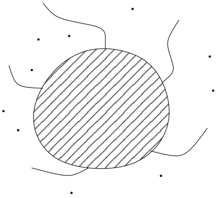
showed that RCC is modelled by the non-empty regular closed subsets of certain topological spaces.

Stell 2000

reformulated models of RCC as Boolean connection algebras (B, C), where C is a binary relation on Boolean algebra B.

$RC(X)$: the BA of regular closed subsets of space X

a is regular closed if $a = \mathrm{cl}(\mathrm{int}(a))$



$$\begin{aligned} a + b &= a \cup b \\ a \cdot b &= \mathrm{cl}(\mathrm{int}(a \cap b)) \\ -a &= \mathrm{cl}(X \backslash a) \\ 0 &= \emptyset \\ 1 &= X \end{aligned}$$

In $RC(\mathbb{R})$, $[0,1] \cdot [1,2] = 0$ $-[0,\infty) = (-\infty, 0]$

BCA: Boolean contact algebra $A = (B_A, C_A)$

Axioms for the contact relation

C1. xCy implies $x, y \neq 0$.

C2. xCy implies yCx.

C3. $xC(y+z)$ iff xCy or xCz.

C4. $x \neq 0$ implies xCx.

Example

$(RC(X), C_X)$, aC_Xb iff $a \cap b \neq \emptyset$, the intersect relation.

Dimov and Vakarelov 2006:

Every BCA is isomorphically embeddable into $(RC(X), C_X)$ for some X that is compact, semiregular and T_0.

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Analogy: The Stone Equivalence

$$\mathbf{BA} \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \mathbf{Stone}$$

The category **BA** of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category **Stone** of Stone spaces and continuous functions.

Stone space := compact, Hausdorff 0-dimensional space

$\Theta \circ \Phi$ and $\Phi \circ \Theta$ are naturally isomorphic to the identity functors

What is the analogue of **Stone** for BCA’s ?

The category **BCA** of Boolean contact algebras and BCA-homomorphisms (?) is dually equivalent to **??**

Answer:
the category of T_0 mereocompact mereotopological spaces and mereomorphisms.

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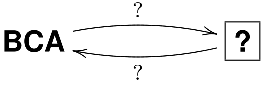

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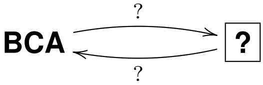

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Mereotopological space $S = (X_S, M_S)$

- X_S is a topological space.
- M_S is a subalgebra of the Boolean algebra $RC(X_S)$.
- M_S is a closed basis for the topology of X_S.

Mereomorphism $\theta : (X_1, M_1) \to (X_2, M_2)$:

a function $\theta : X_1 \to X_2$ whose pullback action $a \mapsto \theta^{-1}(a)$ is a Boolean algebra homomorphism $M_2 \to M_1$.

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Representation by clans

A clan of $A = (B_A, C_A)$ a non-empty $\Gamma \subseteq B_A$ such that:

K1. $0 \notin \Gamma$.

K2. $x \in \Gamma$ and $x \leq y$ implies $y \in \Gamma$.

K3. $x + y \in \Gamma$ implies $x \in \Gamma$ or $y \in \Gamma$.

K4. $x, y \in \Gamma$ implies xCy.

Example: Point clans

In any mereotopological S, $\rho_S(x) = \{a \in M_S : x \in a\}$ is a clan of M_S. So is $\rho_S(x) \cup \rho_S(y)$.

Example:

Any ultrafilter of a BCA, or any union of ultrafilters.

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Mereocompactness

S is mereocompact if

For every $\Gamma, \Delta \subseteq M_S$ with Γ a clan of M_S,

$$\bigcap \Gamma \subseteq \bigcup \Delta \text{ implies } \Gamma \cap \Delta \neq \emptyset.$$

Theorem

Every mereocompact space S is compact.

Proof.

Let $M \subseteq M_S$ have the finite intersection property.
Extend M to an ultrafilter U on the powerset algebra of X_S.
Then $\Gamma = U \cap M_S$ is a clan of M_S.
Put $\Delta = \emptyset$: have $\Gamma \cap \Delta = \emptyset$.
Hence by mereocompactness, $\bigcap \Gamma \not\subseteq \bigcup \Delta = \emptyset$.
Thus $\bigcap M \neq \emptyset$. □

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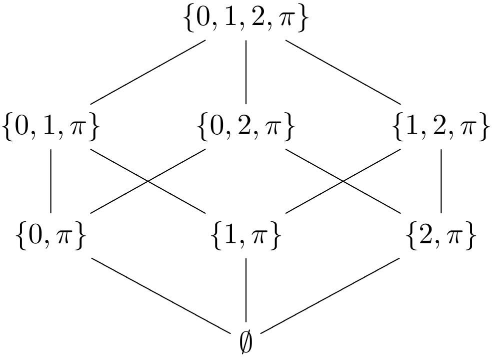

Mereocompactness implies that every clan is a point clan.
This fails for the clan $\Gamma = M_S - \{\emptyset, \{2, \pi\}\}$.

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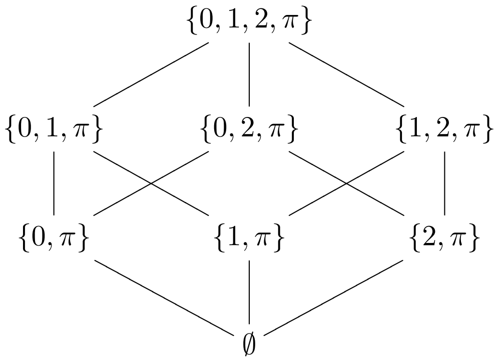

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Variations on the theme:

μ_0: (equivalent to mereocompactness)

For every $\Gamma, \Delta \subseteq M_S$ with Γ a clan of S^+, if $\bigcap \Gamma \subseteq \bigcup \Delta$ then there exists a $\gamma \in \Gamma$ and a finite $\Delta_0 \subseteq \Delta$ such that $\gamma \subseteq \bigcup \Delta_0$.

μ_1: (weaker than μ_0)

For every $\Gamma, \Delta \subseteq M_S$ with $\bigcap \Gamma \subseteq \bigcup \Delta$, there exist finite sets $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ such that $\bigcap \Gamma_0 \subseteq \bigcup \Delta_0$.

If M_S is the dual algebra of a Stone space X_S, then μ_1 is equivalent to the compactness of X_S.

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Reference:

Robert Goldblatt and Matt Grice,
Mereocompactness and Duality for Mereotopological Spaces,

to appear.