Question 1 solutions (2016 Q3)

(a) Let $f(x) = x - a$.

$$a = \int_0^{\frac{\pi}{2}} f(x)\sin x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} (x-a)\sin x\,\mathrm{d}x = [-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + [a\cos x]_0^{\frac{\pi}{2}} = 1 - a$$

$$a = \frac{1}{2} \quad \therefore f(x) = x - \frac{1}{2}$$

(b)

(1) $$\frac{d(e^{2x}y)}{dx} = 2e^{2x}y + e^{2x}\frac{dy}{dx} = 2e^{2x}y + e^{2x}(x-2y) = xe^{2x}$$

(2) $$e^{2x}y = \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

$$e^2 = \frac{1}{2}e^2 - \frac{1}{4}e^2 + c \rightarrow c = \frac{3}{4}e^2$$

$$y = e^{-2x}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + \frac{3}{4}e^2\right) = \frac{1}{4}\left(3e^{2-2x} + 2x - 1\right)$$