Question 2 solutions (2016 Q4)

(a) Asymptotes $y=\pm 3x$, $\tan\alpha = 3$, $\frac{\sin^2\alpha}{1-\sin^2\alpha}=9$, $\sin^2\alpha=\frac{9}{10}$, $\cos^2\alpha=\frac{1}{10}$

$\sin\theta=\sin(180^0-2\alpha)=\sin(2\alpha)=2\times\sqrt{\frac{9}{10}}\times\sqrt{\frac{1}{10}}=0.6$

(b) **Method 1**

$y=\pm 3x$, $\frac{|3x_0-y_0|}{\sqrt{10}}\cdot\frac{|3x_0+y_0|}{\sqrt{10}}=\frac{36}{10}=3.6$

Method 2

$l_1: y=3x$,

$l_{PA}: y-y_0=-\frac{1}{3}(x-x_0)\to A:(0.1x_0+0.3y_0,\ 0.3x_0+0.9y_0)$

$PA=\sqrt{0.9x_0^2+0.1y_0^2-0.6x_0y_0}=\frac{1}{\sqrt{10}}|3x_0-y_0|$

$l_2: y=-3x$,

$l_{PB}: y-y_0=\frac{1}{3}(x-x_0)\to B:(0.1x_0-0.3y_0,\ -0.3x_0,+0.9y_0)$

$PB=\sqrt{0.9x_0^2+0.1y_0^2+0.6x_0y_0}=\frac{1}{\sqrt{10}}|3x_0+y_0|$

$PA\cdot PB=\frac{|3x_0-y_0|}{\sqrt{10}}\cdot\frac{|3x_0+y_0|}{\sqrt{10}}=\frac{36}{10}=3.6$

(c) Let $P(x_0,y_0)$, $m_{CD}=k$. Note: $|k|>3$

$l_{CD}: y-y_0=k(x-x_0)$, $y=\pm 3x$

$C:\left(\frac{kx_0-y_0}{k-3},\frac{3(kx_0-y_0)}{k-3}\right)$, $D:\left(\frac{kx_0-y_0}{k+3},\frac{-3(kx_0-y_0)}{k+3}\right)$

Note that $\sin\angle COD=0.6$

$$\text{Area}_{\Delta COD}=\frac{1}{2}\times 0.6\times\sqrt{10}\left(\frac{kx_0-y_0}{k-3}\right)\times\sqrt{10}\left(\frac{kx_0-y_0}{k+3}\right)$$

$$=\frac{3(kx_0-y_0)^2}{(k^2-9)} \qquad (1)$$

Since CP : PD $=1:\lambda$,

$$x_0=\frac{1}{\lambda+1}\left(\lambda\frac{kx_0-y_0}{k+3}+\frac{kx_0-y_0}{k-3}\right)=\frac{kx_0-y_0}{\lambda+1}\left(\frac{\lambda}{k+3}+\frac{1}{k-3}\right),$$

$$y_0=\frac{1}{\lambda+1}\left(\lambda\frac{-3(kx_0-y_0)}{k+3}+\frac{3(kx_0-y_0)}{k-3}\right)=\frac{3(kx_0-y_0)}{\lambda+1}\left(\frac{-\lambda}{k+3}+\frac{1}{k-3}\right) \qquad (2)$$

$$\frac{x_0^2}{4}-\frac{y_0^2}{36}=1\to\frac{(kx_0-y_0)^2}{(k^2-9)}=\frac{(\lambda+1)^2}{\lambda}$$

$$\text{area}_{\Delta COD}=\frac{3(\lambda+1)^2}{\lambda}=3\left(\lambda+2+\frac{1}{\lambda}\right),$$

min $\text{area}_{\Delta COD}$ = 12, when $\lambda=1$ (since $\lambda+\frac{1}{\lambda}\geq 2$).