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Sonoluminescence: two-photon correlations as a test of thermality

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Abstract

In this Letter we propose a fundamental test for probing the thermal nature of the spectrum emitted by sonoluminescence. We show that two-photon correlations can in principle discriminate between real thermal light and the pseudo-thermal squeezed-state photons typical of models based on the dynamic Casimir effect. Two-photon correlations provide a powerful experimental test for various classes of sonoluminescence models. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this Letter we propose a fundamental test for experimentally discriminating between various classes of theoretical models for sonoluminescence. It is well known that the optical photons measured in sonoluminescence are characterized by a broadband spectrum, often described as approximately thermal

with a ‘temperature’ of several tens of thousands of Kelvin [1]. Whether or not this ‘temperature’ represents an actual thermal ensemble is less than clear. For instance, according to the ‘‘shock wave approach’’ of Barber et al., or the ‘‘adiabatic heating hypothesis’’, thermality of the spectrum is due to a high physical temperature caused by compression of the gases contained in the bubble. On the other hand, in models based on variants of Schwinger’s ‘‘dynamical Casimir approach’’ [2–7], it is possible to avoid reaching high physical temperatures and yet to obtain a thermal spectrum (or at least pseudo-thermal characteristics for the emitted photons) because of the peculiar statistical properties of the two-photon

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squeezed-states produced by this class of mechanism.

We stress that thermal characteristics in single photon measurements can be associated with *at least* two hypotheses: (a) real physical thermalization of the photon environment; (b) pseudo-thermal single photon statistics due to tracing over the unobserved member of a photon pair that is actually produced in a two-mode squeezed state. We shall call case (a) *real thermality*; while case (b) will be denoted *effective thermality*. Of course, case (b) has no relation with any concept of thermodynamic temperature, though to any such squeezed state one may assign a (possibly mode-dependent) *effective temperature*.

Our aim is to find a class of measurements able to discriminate between cases (a) and (b), and to understand the origin of the roughly thermal spectrum for sonoluminescence in the visible frequency range. In principle, the thermal character of the experimental spectrum could disappear at higher frequencies, but for such frequencies the water medium is opaque, and it is not clear how we could detect them. (Except through heating effects.) Our key remark is that it is not necessary to try to measure higher than visible frequencies in order to get a definitive answer regarding thermality. It is sufficient, at least in principle, to measure photon pair correlations in the visible portion of the sonoluminescence spectrum. Thus regardless of the underlying mechanism, two-photon correlation measurements are a very useful tool for discriminating between broad classes of theory and thereby investigating the nature of sonoluminescence. We note that two-photon correlations have already been proposed, for the first time in [8] and subsequently in [9,10], as an efficient tool for measuring the shape and the size of the emission region. It was proposed in [8,9] that precise Hanbury–Brown–Twiss interferometry measurements could in principle distinguish between chaotic (thermal) light emerging from a hot bubble and the possible production of coherent light via the dynamical Casimir effect. Unfortunately in the dynamical Casimir effect photons are always pair-produced from the vacuum in two-mode squeezed states, not in coherent states. Pair-production via the dynamical Casimir effect appears to imply that all the photon pairs form two-mode squeezed states, which are very different from the coherent states analyzed in [8–10].

2. Real thermal light versus two-mode squeezed states

The quantum optics mechanism that simulates a thermal spectrum [case (b)] is based on two-mode squeezed-states defined by

$$|\zeta_{ab}\rangle = e^{-\zeta(a^\dagger b^\dagger - ba)}|0_a, 0_b\rangle, \quad (1)$$

where ζ is (for our purposes) a real parameter though more generally it can be chosen to be complex [11]. In quantum optics a two-mode squeezed-state is typically associated with a so called non-degenerate parametric amplifier (one of the two photons is called “signal” and the other “idler” [11–13]). Consider the operator algebra

$$[a, a^\dagger] = 1 = [b, b^\dagger], \quad [a, b] = 0 = [a^\dagger, b^\dagger], \quad (2)$$

and the corresponding vacua

$$|0_a\rangle: a|0_a\rangle = 0, \quad |0_b\rangle: b|0_b\rangle = 0. \quad (3)$$

The two-mode vacuum is the state $|\zeta\rangle \equiv |0(\zeta)\rangle$ annihilated by the operators

$$A(\zeta) = \cosh(\zeta) a - \sinh(\zeta) b^\dagger, \quad (4)$$

$$B(\zeta) = \cosh(\zeta) b - \sinh(\zeta) a^\dagger. \quad (5)$$

A characteristic of two-mode squeezed states is that if we measure only one photon and “trace away” the second, a thermal density matrix is obtained [11–13]. Indeed, if O_a represents an observable relative to one mode (say mode ‘a’) its expectation value on the squeezed vacuum is given by

$$\begin{aligned} \langle \zeta_{ab} | O_a | \zeta_{ab} \rangle \\ = \frac{1}{\cosh^2(\zeta)} \sum_{n=0}^{\infty} [\tanh(\zeta)]^{2n} \langle n_a | O_a | n_a \rangle. \end{aligned} \quad (6)$$

In particular, if we consider $O_a = N_a$, the number operator in mode a , the above reduces to

$$\langle \zeta_{ab} | N_a | \zeta_{ab} \rangle = \sinh^2(\zeta). \quad (7)$$

These formulae have a strong formal analogy with thermofield dynamics (TFD) [14,15], where a doubling of the physical Hilbert space of states is invoked in order to be able to rewrite the usual Gibbs (mixed state) thermal average of an observable as an expectation value with respect to a temperature-dependent ‘vacuum’ state (the thermofield vacuum, a pure state in the doubled Hilbert space). In the TFD

approach, a trace over the unphysical (fictitious) states of the fictitious Hilbert space gives rise to thermal averages for physical observables, completely analogous to the averages in Eq. (6) *except* that we must make the following identification

$$\tanh(\zeta) = \exp\left(-\frac{1}{2} \frac{\hbar \omega}{k_B T}\right), \quad (8)$$

where ω is the mode frequency and T is the temperature. We note that the above identification implies that the squeezing parameter ζ in TFD is ω -dependent in a very special way.

The formal analogy with TFD allows us to conclude that, provided we measure only one photon mode, the two-mode squeezed-state acts as a thermal vacuum and the single-mode expectation values acquire a pseudo-thermal character corresponding to a ‘temperature’ $T_{\text{squeezing}}$ related with the squeezing parameter ζ by [13]

$$k_B T_{\text{squeezing}} = \frac{\hbar \omega_i}{2 \log(\coth(\zeta))}, \quad (9)$$

where the index $i = a, b$ indicates the signal mode or the idler mode respectively; note that ‘signal’ and ‘idler’ modes can have different effective temperatures (in general $\omega_{\text{signal}} \neq \omega_{\text{idler}}$) [13].

3. A toy model and sonoluminescence

To treat sonoluminescence, we introduce a quantum field theory characterized by an infinite set of bosonic oscillators (as in bosonic TFD; not just two oscillators as in the case of ‘signal–idler’ systems studied in quantum optics). The simple two-mode squeezed vacuum is replaced by

$$|\Omega[\zeta(k, k')]\rangle \equiv \exp\left[-\int d^3k d^3k' \zeta(k, k') \times (a_k b_{k'} - a_k^\dagger b_{k'}^\dagger)\right] |0\rangle, \quad (10)$$

where the function $\zeta(k, k')$ is peaked near $k + k' = 0$, and becomes proportional to a delta function in the case of infinite volume [$\zeta(k, k') \rightarrow \zeta(k) \delta(k + k')$] when the photons are emitted strictly back-to-back [16–19]. To be concrete, let us refer to the homogeneous dielectric model presented in [18]. In this limit

there is no ‘mixing’ and everything reduces to a sum of two-mode squeezed-states, where each pair of back-to-back modes is decoupled from the other. The frequency ω is the same for each photon in the couple, in such a way that we are sure to get the same ‘temperature’ for both. The two-mode squeezed vacuum then simplifies to

$$|\Omega(\zeta_k)\rangle \equiv \exp\left[-\int d^3k \zeta_k (a_k a_{-k} - a_k^\dagger a_{-k}^\dagger)\right] |0\rangle. \quad (11)$$

The key to the present proposal is that, if photons are pair produced via the dynamical Casimir effect, then they are actually produced in some combination of these two-mode squeezed-states [16–19]. In this case $T_{\text{squeezing}}$ is a function of both frequency and squeezing parameter, and in general only a special ‘fine tuning’ would allow us to get the *same* effective temperature for all couples. If we consider the expectation value on the state $|\Omega(\zeta_k)\rangle$ of $N_k \equiv a_k^\dagger a_k$ we get

$$\langle \Omega(\zeta_k) | N_k | \Omega(\zeta_k) \rangle = \sinh^2(\zeta_k), \quad (12)$$

so we again find a ‘thermal’ distribution for each value of k with temperature

$$k_B T_k \equiv \frac{\hbar \omega_k}{2 \log(\coth(\zeta_k))}. \quad (13)$$

The point is that for $k \neq \bar{k}$ we generally get $T_k \neq T_{\bar{k}}$ *unless a fine tuning condition holds*. This condition is implicitly enforced in the definition of the thermal vacuum and it is possible only if we have

$$\coth(\zeta_k) = e^{\kappa \omega_k}, \quad (14)$$

with κ some constant, so that the frequency dependence in T_k is canceled and the same $T_{\text{squeezing}}$ is obtained for all couples.

For models of sonoluminescence based on the dynamical Casimir effect (i.e. squeezing the QED vacuum) we cannot rely on a definition to provide the fine tuning, but must perform an actual calculation. Our model [18] is again a useful tool for a quantitative analysis. We have (omitting indices for notational simplicity; our Bogolubov transformation is diagonal) the following relation between the squeezing parameter and the Bogolubov coefficient β

$$\langle N \rangle = \sinh^2(\zeta) = |\beta|^2. \quad (15)$$

By defining $\tau \equiv \pi t_0 / (n_{\text{in}}^2 + n_{\text{out}}^2)$, where t_0 is the timescale on which the refractive index changes, one has [18]

$$|\beta(\mathbf{k}_1, \mathbf{k}_2)|^2 = \frac{\sinh^2(|n_{\text{in}}^2 \omega_{\text{in}} - n_{\text{out}}^2 \omega_{\text{out}}| \tau)}{\sinh(2 n_{\text{in}}^2 \omega_{\text{in}} \tau) \sinh(2 n_{\text{out}}^2 \omega_{\text{out}} \tau)} \times \frac{V}{(2\pi)^3} \delta^3(\mathbf{k}_1 + \mathbf{k}_2). \quad (16)$$

In the adiabatic limit (large frequencies) we get a Boltzmann factor [18]

$$|\beta|^2 \approx \exp(-4 \min\{n_{\text{in}}, n_{\text{out}}\} n_{\text{out}} \omega_{\text{out}} \tau). \quad (17)$$

Since $|\beta|$ is small, $\sinh(\zeta) \approx \tanh(\zeta)$, so that in this adiabatic limit

$$|\tanh(\zeta)|^2 \approx \exp(-4 \min\{n_{\text{in}}, n_{\text{out}}\} n_{\text{out}} \omega_{\text{out}} \tau). \quad (18)$$

Therefore

$$k_B T_{\text{effective}} \approx \frac{\hbar}{8\pi t_0} \frac{n_{\text{in}}^2 + n_{\text{out}}^2}{n_{\text{out}} \min\{n_{\text{in}}, n_{\text{out}}\}}. \quad (19)$$

Thus for the entire adiabatic region we can assign a *single* frequency-independent effective temperature, which is really a measure of the speed with which the refractive index changes. Physically, in sonoluminescence this observation applies only to the high-frequency tail of the photon spectrum.

In contrast, in the low frequency region, where the bulk of the photons emitted in sonoluminescence are to be found, the sudden approximation holds and the spectrum is phase-space-limited (a power law spectrum), not Planckian [18]. It is nevertheless still possible to assign a *different* effective temperature for each frequency.

Finite volume effects smear the momentum-space delta function so we no longer get exactly back-to-back photons. This represents a further problem because we have to return to the general squeezed vacuum of Eq. (10). It is still true that photons are emitted in pairs, pairs that are now approximately back-to-back and of approximately equal frequency. We can again define an effective temperature for each photon in the couple as in the ‘signal–idler’ systems of quantum optics. This effective temperature is no longer the same for the two photons belonging to the same couple and no ‘special condi-

tion’ for getting the same temperature for all the couples exists. Hence the analysis of these finite volume distortions is not easy [19], but the qualitative result that in any dynamic Casimir effect model of sonoluminescence there should be strong correlations between approximately back-to-back photons is robust.

Indeed, if we work with a plane wave approximation for the electromagnetic eigen-modes (this is essentially a version of the Born approximation, modified to deal with Bogolubov coefficients instead of scattering amplitudes) and further modify the infinite-volume model of [18], both by permitting a more general temporal profile for the refractive index, and by cutting off the space integrations at the surface of the bubble, then the squared Bogolubov coefficient takes the form

$$|\beta(\mathbf{k}_1, \mathbf{k}_2)|^2 = F(k_1, k_2; n(t)) |S(|\mathbf{k}_1 + \mathbf{k}_2| R)|^2. \quad (20)$$

Here $F(k_1, k_2; n(t))$ is some complicated function of the refractive index temporal profile, which encodes all the dynamics, while $S(|\mathbf{k}_1 + \mathbf{k}_2| R)$ is a purely kinematical factor arising from the limited spatial integration:

$$S(|\mathbf{k}_1 + \mathbf{k}_2| R) \equiv \int_{r \leq R} d^3 \mathbf{r} \exp[-i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}].$$

Indeed in the infinite volume limit $|S(\mathbf{k}_1, \mathbf{k}_2)|^2 \rightarrow [V/(2\pi)^3] \delta(\mathbf{k}_1 + \mathbf{k}_2)$. It is now a standard calculation to show that

$$S(|\mathbf{k}_1 + \mathbf{k}_2| R) = \frac{4\pi}{|\mathbf{k}_1 + \mathbf{k}_2|^3} [\sin(|\mathbf{k}_1 + \mathbf{k}_2| R) - (|\mathbf{k}_1 + \mathbf{k}_2| R) \cos(|\mathbf{k}_1 + \mathbf{k}_2| R)].$$

So, independent of the temporal profile, kinematics will provide characteristic angular correlations between the outgoing photons: this result depends only on the the existence of a vacuum squeezing effect driven by a time-dependent refractive index (which is what is needed to make the notion of a Bogolubov coefficient meaningful in this context).

The plane-wave approximation used to obtain this formula is valid provided the wavelength of the photons, *while they are still inside the bubble*, are small compared to the dimensions of the bubble

$$\lambda_{\text{inside}} \ll R; \quad \Rightarrow \quad \omega \gg \frac{c}{nR}. \quad (21)$$

While there is still considerable disagreement about the physical size of the bubble when light emission occurs [17], and almost no data concerning the value of the refractive index of the bubble contents at that time, the scenario developed in [18,19] is very promising in this regard. In particular, high frequency photons are more likely to exhibit the back-to-back effect, and depending on the values of R and n this could hold for significant portions of the resulting emission spectrum. Experimentally, one should work at as high a frequency as possible – at the peak close to the cutoff.

These observations lead us to the following proposal.

4. Two-photon observables

Define the observable

$$N_{ab} \equiv N_a - N_b, \quad (22)$$

and its variance

$$\Delta(N_{ab})^2 = \Delta N_a^2 + \Delta N_b^2 - 2\langle N_a N_b \rangle + 2\langle N_a \rangle \langle N_b \rangle. \quad (23)$$

These number operators N_a, N_b are intended to be relative to photons measured, e.g., back to back. In the case of true thermal light we get

$$\Delta N_a^2 = \langle N_a \rangle (\langle N_a \rangle + 1), \quad (24)$$

$$\langle N_a N_b \rangle = \langle N_a \rangle \langle N_b \rangle, \quad (25)$$

so that

$$\begin{aligned} \Delta(N_{ab})_{\text{thermal light}}^2 \\ = \langle N_a \rangle (\langle N_a \rangle + 1) + \langle N_b \rangle (\langle N_b \rangle + 1). \end{aligned} \quad (26)$$

For a two-mode squeezed-state

$$\Delta(N_{ab})_{\text{two mode squeezed light}}^2 = 0. \quad (27)$$

Due to correlations, $\langle N_a N_b \rangle \neq \langle N_a \rangle \langle N_b \rangle$. Note also, that if you measure only a single photon in the couple, you get (as expected) a thermal variance $\Delta N_a^2 = \langle N_a \rangle (\langle N_a \rangle + 1)$. Therefore a measurement of the covariance $\Delta(N_{ab})^2$ can be decisive in discriminating if the photons are really physically thermal or if non classical correlations between the photons occur [12]. If the ‘thermality’ in the sonoluminescence spectrum is of this squeezed-mode type,

we will ultimately desire a much more detailed model of the dynamical Casimir effect involving an interaction term that produces pairs of photons in two-mode squeezed-states. Apart from our model [18] and its finite volume generalization [19], the Eberlein model also possesses this property [20–23]. For this type of squeezed-mode photon pair-production in a linear medium with spacetime-dependent dielectric permittivity and magnetic permeability see [24]; for nonlinearity effects see [25].

In summary: The main experimental signature for squeezed-state photons being pair-produced in sonoluminescence is the presence of strong spatial correlations between photons emitted back-to-back and having the same frequency. These correlations could be measured, for example, by back-to-back symmetrically placed detectors working in coincidence. Finite-size effects have been shown in [19] to perturb only slightly this back-to-back character of the emitted photons, in the sense that back-to-back emission remains largely dominant. (Additionally it has been verified that the form of the spectrum is not violently affected.) Of course, a detailed analysis of the many technical experimental problems (such as e.g. filtering and multi-mode signals in the detectors) has also to be done (on these topics see [8]), but such technical details are beyond the scope of the current work.

5. Discussion

The main aims of the present Letter are to clarify the nature of the photons produced in Casimir-based models of sonoluminescence, and to delineate the available lines of (theoretical as well experimental) research that should be followed in order to discriminate Casimir-based models from thermal models, preferably without having to understand all of the messy technical details of the condensed matter physics taking place inside the collapsing bubble.

We have shown that ‘effective thermality’ can manifest itself at different levels. What is certainly true is that two-mode squeezed states will exhibit, at a given fixed three-momentum, occupation numbers which in that mode follow Bose–Einstein statistics. This can be called ‘‘thermality at fixed wave-number’’. In contrast, it is sometimes possible to

assign, at least for a reasonably wide range of wavenumbers, the *same* temperature to all modes. This “thermality across a range of wavenumbers” gives rise, at least in this range of wavenumbers, to a spectrum which is approximately Planckian.

Our sonoluminescence model exhibits Bose–Einstein thermality but not a truly Planckian spectrum (since the bulk of the photon emission occurs at frequencies where the sudden approximation holds and a common temperature for all the momenta is lacking). The spectrum is generically a power law at low frequencies followed by a cut-off [18,19]. Although precise measurements in the low frequency tail of the spectrum could also (in principle) allow us to discriminate class ‘a’ models from class ‘b’ models, this possibility has to be considered strongly model-dependent. Furthermore the spectral data available at the present time is in this regard relatively crude: spectral analysis by itself does not seem to be an appropriate tool for discriminating between class ‘a’ and class ‘b’ models.

Despite this limitation we have shown that there is still the possibility of obtaining a clear discrimination between real and effective thermality, without relying on the detailed features of the model, by looking at two-photon correlations. For thermal light one should find thermal variance for photon pairs. On the other hand, thermofield-like photons should show zero variance in appropriate pair correlations. Moreover, our analysis points out that a key point in discriminating, by means of photon measurements alone, between classes of models for sonoluminescence is the mechanism of photon production: Any form of pair-production is associated with two-mode squeezed states and their strong quantum correlations. In contrast, any single-photon production mechanism (thermal, partially thermal, non-thermal) is not. In either case, two-photon correlation measurements are potentially a very useful tool for looking into the nature of sonoluminescence.

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