Modified gravity: why and how?

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Historical time-line

- 16th century: Galileo Galilei uses pendulums and inclined planes to test terrestrial gravity (pre-history!)
- 1665: Sir Isaac Newton introduces the inverse square law - terrestrial and celestial gravity united
- 1855: Urbain Le Verrier observes a 35 arc-second excess precession of Mercury's orbit.
- 1882: Simon Newcomb measure this precession to be 43 arc-seconds. Newtonian gravity fails to explain it!
- 1893: Ernst Mach states "Mach's Principle"
- 1905: Albert Einstein introduces Special Relativity - theoretical clash with Galilean ideas on relative motion.
- 1915: Albert Einstein completes the General Theory of Relativity (GR). The theory explains Mercury's precession.
- 1919: Arthur Eddington measures light deflection during a Solar eclipse and verifies the prediction of GR

Newton vs Einstein

By 1905 Newtonian gravity faces serious challenges:

- Experimental: Mercury's precession
- Theoretical: Clash with Special Relativity

Result

By 1915 Newton's theory is replaced by General Relativity

Is becoming obsolete the fate of any physical theory?
Is GR bound to face the same challenges?

A high-energy theory of gravity?

Fact

GR is a classical theory $\rightarrow$ what happens in the Plank scale where things turn quantum?

Reason to fit together GR and Quantum Field Theory (QFT):

- searching for the unknown
- intrinsic limit of GR and QFT
- conceptual clash
- vision for unification

Cosmological Riddle no. 1: inflation

Motivation:

- Horizon problem: how did the universe become so homogeneous?
- Flatness problem: how did the universe become so (spatially) flat?

Proposed solution (1980s, Guth and others)

A period of (quasi)-exponential expansion

All length will rapidly become larger than the Hubble radius and Ω_k is driven to 0.

Cosmological Riddle no. 2: late-time observations

- Cosmic Microwave Background: COBE, Toco, BOOMERanG, MAXIMA, WMAP
- Galaxy distribution surveys and Supernovae: 2dF GRS, SDSS, SLS

Combined Results

- $\Omega_k = -0.015^{+0.020}_{-0.016}$
- $\Omega_m \sim 0.24$
- $\Omega_b \sim 0.04$
- $\Omega - \Omega_k - \Omega_m \sim 0.76$

20% dark matter and 76% dark energy and on top of that $w_{de} = -1.06^{+0.13}_{-0.08}$!!!

Astrophysical Observations

1. 1933: "Missing Mass" question posed by Fritz Zwicky (Coma Cluster- Virial mass$\sim$ 400$\times$Galaxy count mass)
2. 1959: Kahn and Waljter propose the presence of dark matter in individual galaxies
3. 1970's: flat rotational curves (Rubin, Bosma)

Dark matter appears to be in Galaxies and Galaxy clusters as well as in Cosmology.

The concordance model (ΛCDM)

Ingredients:

1. General Relativity
2. Cold Dark Matter (20%)
3. Cosmological Constant (76% - $\rho_\Lambda \sim (10^{-3}\mathrm{eV})^4$)

Pros

Fits the data with just one parameter

Cons

Poor theoretical motivation

- Cosmological constant problem
- Coincidence problem

Scalar fields in cosmology

Energy density and Pressure:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

For an effective EOS $p_\phi = w_\phi \rho_\phi$

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \to -1$$

when $V(\phi) \gg \dot{\phi}^2$. Can lead to acceleration!

Typical example

The inflaton field: scalar field that drives inflation

Dark Energy Problem 1: Vacuum Energy

Empty space can have a non-zero energy density:

$$< T_{\mu\nu} >= - < \rho > g_{\mu\nu}, \qquad < \rho > \neq 0$$

Example: scalar field

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla_{\rm sp}\phi)^2 + V(\phi)$$

ρ_ϕ is constant for any constant ϕ_0 but $V(\phi_0) \neq 0$

Therefore

$$\Lambda = 8\pi G < \rho >$$

Dark Energy Problem 1: Vacuum Energy

- Observations:

$$\rho_\Lambda \sim (10^{-3}\mathrm{eV})^4$$

- Theory with Planck scale cutoff:

$$\rho_\Lambda \sim (10^{27}\mathrm{eV})^4$$

- Theory with SUSY scale cutoff:

$$\rho_\Lambda \sim M^4_{\mathrm{SUSY}} \geq (10^{12}\mathrm{eV})^4$$

Worst theoretical prediction ever!

Dark Energy Problem 2: Anthropic Principle

Weak Anthropic Principle

Observers will only observe conditions which allow for observers

Moral: If the cosmological constant had another value we would be here to talk about it!

Related ideas

- Multiverse
- String Landscape scenario

Dark Energy Problem 3: Quintessence

Scalar field, much like the inflaton, leads to late time acceleration $(V(\phi) \gg \dot{\phi}^2)$

Pros

- Simplicity and familiarity
- Tracker solutions may solve the coincidence problem

Cons

- Poor theoretical motivation $(m_\phi \sim 10^{-33}$ eV)
- Significant fine tuning

Dark Matter Problem

Nature of Dark matter

- Baryonic: brown dwarfs, massive black holes, cold diffuse gas [minimal contribution]
- Non baryonic: hot dark matter (neutrino), cold dark matter (lots) [main contribution]

CDM Candidates:

- axion
- neutralino
- sneutrino
- gravitino
- axino
- ...

Quantum Gravity candidates

String Theory:

- Building blocks: one dimensional strings
- Attempt for unification
- Different resonant frequencies lead to different forces

Loop Quantum Gravity:

- Direct quantization: loop quantization
- A theory of quantum gravity - no unification

Remark

No current theory reduces to GR at the classical limit!

Maybe we should reconsider about GR?

Similar problems that Newton's Theory faced:

- Theoretical: Clash with Quantum Field theory
- Observational: Inflation, late-time acceleration, dark matter etc.

Proposed way out

Alternative theory of gravitation which:

1. Comes as a low energy limit of a more fundamental theory
2. Includes infrared corrections with respect to GR
3. Can account for some or all of the unexplained observations

Describing spacetime

Spacetime: 4-manifold equipped with

- metric, $g_{\mu\nu}$: measurement of distances → dot product
- connection, $\Gamma^{\lambda}{}_{\mu\nu}$: parallel transport → covariant derivative

$$\bar{\nabla}_\mu A^\nu{}_\sigma = \partial_\mu A^\nu{}_\sigma + \Gamma^\nu{}_{\mu\alpha} A^\alpha{}_\sigma - \Gamma^\alpha{}_{\mu\sigma} A^\nu{}_\alpha$$

Curvature: Riemann Tensor

$$\mathcal{R}^\mu{}_{\nu\sigma\lambda} = -\partial_\lambda \Gamma^\mu{}_{\nu\sigma} + \partial_\sigma \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\lambda} - \Gamma^\mu{}_{\alpha\lambda}\Gamma^\alpha{}_{\nu\sigma}.$$

Non-metricity tensor and Cartan torsion tensor:

$$Q_{\mu\nu\lambda} \equiv -\bar{\nabla}_\mu g_{\nu\lambda} \qquad\qquad S_{\mu\nu}{}^{\lambda} \equiv \Gamma^\lambda{}_{[\mu\nu]}$$

General Relativity in a nutshell

Assumptions

1. $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}$ or $S_{\mu\nu}{}^{\lambda} = 0$. Spacetime is torsion-less.
2. $\bar{\nabla}_{\lambda} g_{\mu\nu} = 0$ or $Q_{\mu\nu\lambda} = 0$. The connection is a metric one.
3. No fields other than the metric mediate gravity.
4. The field equations should be second order PDEs.
5. Covariant field equations or diffeo-invariant action.

Metric Curvature Tensor

$$R^{\mu}{}_{\nu\sigma\lambda} = -\partial_{\lambda}\{^{\mu}{}_{\nu\sigma}\} + \partial_{\sigma}\{^{\mu}{}_{\nu\lambda}\} + \{^{\mu}{}_{\alpha\sigma}\}\{^{\alpha}{}_{\nu\lambda}\} - \{^{\mu}{}_{\alpha\lambda}\}\{^{\alpha}{}_{\nu\sigma}\}.$$

Einstein–Hilbert Action:

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_M(g^{\mu\nu}, \psi)$$

Palatini variation

Abandon the assumption $\bar{\nabla}_\lambda g_{\mu\nu} = 0 \Rightarrow$ Independent connection

$$S_{\rm p} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\mathcal{R} + S_M(g^{\mu\nu}, \psi)$$

Variation with respect to the metric gives

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\, g_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}$$

Variation with respect to the connection gives

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\},$$

General Relativity all over again!

Relaxing other assumptions

- Theories with higher order field equations
  Example:

$$S = \int d^4x \sqrt{-g} L(R, R^{\mu\nu} R_{\mu\nu}) + S_M(g^{\mu\nu}, \psi)$$

- Theories with extra fields mediating gravity
  1. Scalar fields (Nordstöm's theory, Brans-Dicke theory)
  2. Vector fields (TeVeS, Einstein-Aether theory)
  3. Tensor fields (bi-metric-like theories)
  4. Affine connections (Einstein-Cartan theory)

Scalar-tensor theory

Action

$$S_{ST} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}(\partial_\mu\phi\partial^\mu\phi) - V(\phi)\right] + S_M$$

Field equations:

$$G_{\mu\nu} = \frac{8\pi G}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\right) + \\ + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V}{2\phi}g_{\mu\nu}$$

$$(2\omega(\phi)+3)\Box\phi = 8\pi G\, T - \omega'(\phi)\nabla^\lambda\phi\nabla_\lambda\phi + \phi V' - 2V,$$

Metric $f(R)$ gravity

Action

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \psi).$$

Field equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f' = 8\pi G\, T_{\mu\nu}.$$

Trace of the field equations:

$$f'(R)R - 2f(R) + 3\Box f' = 8\pi G\, T$$

Palatini $f(R)$ gravity

Action

$$S_{pal} = \frac{1}{16\pi\, G} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi)$$

Field equations:

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}$$

$$\bar{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) = 0$$

Trace of the first field equation:

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 8\pi\, G\, T$$

Metric-Affine $f(R)$ gravity

Action

$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi)$$

Now S_M has an explicit dependence on $\Gamma^{\lambda}{}_{\mu\nu}$!

Stress-Energy Tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$

"Hypermomentum"

$$\Delta_{\lambda}{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}$$

The connection $\Gamma^{\lambda}{}_{\mu\nu}$ is

- non-metric, *i.e.* $Q_{\mu\nu\lambda} \neq 0$
- not symmetric, *i.e.* $S_{\mu\nu}{}^{\lambda} \neq 0$

$\Delta_\lambda^{\ \mu\nu} \neq 0$, Torsion and Non-metricity

Fact

Angular momentum interacts with geometry

Expectation

Spin should also interact with geometry $\longrightarrow$ torsion

In general

- $\Delta_\lambda^{\ [\mu\nu]} \neq 0$ implies $S_{\mu\nu}^{\ \ \ \lambda} \neq 0$ and matter introduces torsion
- $\Delta_\lambda^{\ (\mu\nu)} \neq 0$ implies that matter also introduces non-metricity

Outcome: a theory with torsion introduced by particles with spin

- Gauge fields (massless particles) will not introduce torsion
- Particles with spin will generally introduce torsion
  - Dirac fields are typical examples

$\Delta_\lambda^{\ \mu\nu} = 0$, Vacuum and Electrovacuum

When $\Delta_\lambda^{\ \mu\nu} = 0$ the theory reduces to Palatini $f(R)$ gravity:

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}$$
$$\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) = 0$$

Example: Electrovacuum where also $T = T^\mu_\mu = 0$ and

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0 \Rightarrow \mathcal{R} = c_i$$

Field equations

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{4}c_i g_{\mu\nu} = \kappa' T^{EM}_{\mu\nu}, \quad \Gamma^\lambda_{\ \mu\nu} = \{^{\lambda}_{\ \mu\nu}\}, \quad \kappa' = \kappa/f'(c_i)$$

Formally equivalent to Einstein equations but $\kappa \neq \kappa'$

$f(R)$ gravity and Brans-Dicke theory

Metric $f(R)$ action

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\psi).$$

Introduce a new field χ and write a dynamically equivalent action:

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(f(\chi) + f'(\chi)(R-\chi)\right) + S_M(g_{\mu\nu},\psi).$$

Redefining χ by $\phi = f'(\chi)$ and setting $V(\phi) = \chi(\phi)\phi - f(\chi(\phi))$

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\phi R - V(\phi)\right) + S_M(g_{\mu\nu},\psi)$$

This is $\omega_0 = 0$ Brans-Dicke theory!

$f(R)$ gravity and Brans-Dicke theory

Palatini $f(R)$ action

$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu}\psi)$$

Following the same approach we get

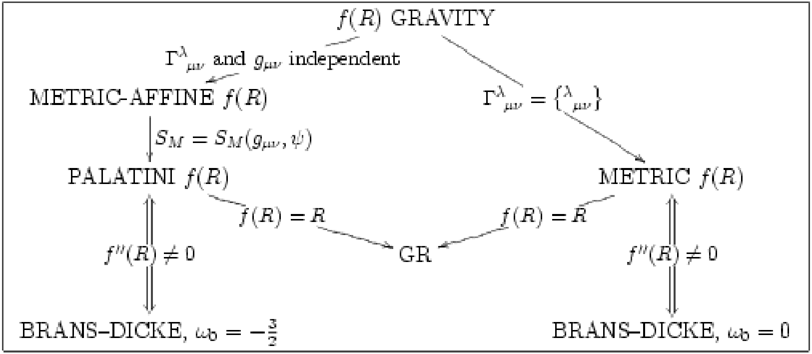
$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(\phi\mathcal{R} - V(\phi)) + S_M(g_{\mu\nu},\psi).$$

Not Brans-Dicke yet! Since $\mathcal{R} = R + \frac{3}{2\phi^2}\nabla_\mu\phi\nabla^\mu\phi - \frac{3}{\phi}\Box\phi$

$$S_{pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\phi R + \frac{3}{2\phi}\nabla_\mu\phi\,\nabla^\mu\phi - V(\phi)\right] + S_M(g_{\mu\nu},\psi)$$

This is $\omega_0 = -3/2$ Brans-Dicke theory!

A classification



Cosmology

General treatment for all theories

- Use the standard cosmological principles
- Assume that the spacetime is described by the FLRW metric
- Insert the metric in the field equations and derive the modified Friedmann equations
- Use the modified Friedmann equations to study the dynamics of the universe

Specific models in all of the previous theories can lead to early and/or late time acceleration!

Example: Metric $f(R)$ gravity

Friedmann equations for a spatially flat FLRW metric

$$H^2 = \frac{1}{3}\kappa\,\rho_{\rm tot} \qquad \frac{\ddot a}{a} = -\frac{\kappa}{6}\left[\rho_{\rm tot} + 3p_{\rm tot}\right]$$

where $\rho_{\rm tot} = \rho + \rho_{de}$, $p_{\rm tot} = p + p_{de}$ and

$$\rho_{de} = \rho_{de}\left[f(R), f'(R), f''(R)\right]$$
$$p_{de} = p_{de}\left[f(R), f'(R), f''(R)\right]$$

A simple model

$$f(R) = R - \mu^4/R \qquad \Rightarrow \qquad w_{de} = -\frac{2}{3}$$

Cosmological constraints

Possible way to impose constraints:

- presence of standard eras
- Stability of the de Sitter solution
- Big Bang Nucleosynthesis
- Perturbations and structure formation

Results so far

- Metric $f(R)$: Stability gives constraints, structure formation and matter era make simple models non-viable
- Palatini $f(R)$: Only models close to the ΛCDM model survive large scale structure formation, power spectrum *etc.*

Nearly Newtonian regime

Metric $f(R)$ gravity:

- Equivalent Brans-Dicke theory: $\omega_0 = 0$
  (experiments: $\omega_0 > 40\,000$)
- $V = Rf' - f$ has to correspond to a large mass for the scalar
- Chameleon behavior
- long debate...

Curvature scalar linear instability

Metric $f(R)$ gravity:

Trace of the field equation

$$3\Box f'(R) + f'(R)R - 2f(R) = 8\pi\, G\, T,$$

Weak field limit, $f(R) = R + \epsilon\varphi(R)$ and $R = -8\pi\, G\, T + R_1$:

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{16\pi\, G\, \varphi'''}{\varphi''}\left(\dot{T}\dot{R}_1 - \vec{\nabla} T \cdot \vec{\nabla} R_1\right) + \frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right) R_1 = 8\pi\, G\, \ddot{T} - 8\pi\, G\, \nabla^2 T - \frac{(8\pi\, G\, T\varphi' + \varphi)}{3\varphi''},$$

An instability occurs if $\varphi'' = f''(R) < 0$ and ϵ is very small

Curvature scalar non-linear instability

Use the redefinition:

$$\varphi = f'(R) - 1, \qquad U'(\phi) = -\frac{1}{3}\left[f'(R)R - 2f(R)\right]$$

Trace of the field equation

$$\Box\varphi - U'(\varphi) = \frac{8\pi}{3} G\, T,$$

Starobinsky's model:

$$f(R) = R + \lambda R_0 \left[\left(1 + \frac{R^2}{R_0^2}\right)^{-n} - 1\right]$$

Nearly Newtonian regime

Palatini $f(R)$ gravity:

- Equivalent Brans-Dicke theory: $\omega_0 = -3/2$ and usual PPN expansion does not apply!
- Existence and uniqueness of Schwarzschild-de Sitter solution lead to misconceptions
- Long debate again...
- Results are density dependent!
- PPN metric is algebraically related to the density!

Example

$$h^1_{00}(t,x) = 2G_{\rm eff}\frac{M_0}{r} + \frac{V_0}{6\phi_0}r^2 + \Omega(\rho)$$

Palatini $f(R)$ gravity and conflict with particle physics

In Palatini $f(R)$ gravity matter is minimally coupled to the metric.

However...

- consider some matter field, *e.g.* a Dirac field (Flanagan) or a Higgs field (Inglesias *et al*), at the local frame
- Recall that ϕ is algebraically related to the matter in the Brans-Dicke representation!
- Perturbative treatment of ϕ breaks down

Outcome

Non-perturbative corrections and strong coupling at low energies!

Non-vacuum solutions in Palatini $f(R)$

Consider static spherically symmetric polytropes ($p = k\rho_0^\Gamma$)

Result

Matching with unique Schwarzschild-de Sitter exterior leads to singularity on the surface for $3/2 < \Gamma < 2$ and practically independent of the functional form of f!

Idealised situation but physically relevant:

- isentropic monoatomic gas ($\Gamma = 5/3$)
- degenerate non-relativistic gas ($\Gamma = 5/3$)

Cause: Lack of dynamics, algebraic dependence of the metric on the matter fields

Non-cumulativity as the root of all evil!

Solving for the connection and replacing back gives

Field Equations

$$G_{\mu\nu} = \frac{8\pi G}{f'} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\left(\mathcal{R} - \frac{f}{f'}\right) + \frac{1}{f'}\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) f' - \\ - \frac{3}{2}\frac{1}{f'^2}\left((\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2} g_{\mu\nu} (\nabla f')^2\right),$$

- single field equation
- second order in the metric — higher order in the matter fields
- non-cumulativity responsible for problems in PPN limit, singularities, disagreement with particle physics, Cauchy problem

Summary and comments

- "Easy" to handle theories were examined to study the difficulties and limitations of modified gravity
- We have learned a lot on the theoretical side: variational principles, importance of couplings, equivalence of theories...
- Interesting perspectives on the observational side: simple models can account for unexplained observations

However...

- Difficulties in solving problems without creating new ones
- Hard to construct a theory that does well at all scales
- Proposing and testing theories is quite a tedious task!
- Could we bootstrap our way to the correct theory from principles?

Projective invariance

$$\Gamma^{\lambda}{}_{\mu\nu} \to \Gamma^{\lambda}{}_{\mu\nu} + \delta^{\lambda}{}_{\mu}\xi_{\nu},$$

where ξ_ν is an arbitrary covariant vector field. Then

$$\mathcal{R} \to \mathcal{R},$$

but matter is not invariant under projective transformations!

Suggested solution

Use a mild constraint on the connection to break this invariance

Lagrange multiplier, B^μ, for this purpose

$$S_{LM} = \int d^4x \sqrt{-g} B^\mu S_\mu.$$

will lead to $S_\mu = S_{\sigma\mu}{}^{\sigma} = 0$.

Field equations

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu},$$

$$\frac{1}{\sqrt{-g}}\left[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu{}_\lambda\right] +$$
$$+2f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu} = \kappa(\Delta_\lambda{}^{\mu\nu} - \frac{2}{3}\Delta_\sigma{}^{\sigma[\nu}\delta^{\mu]}{}_\lambda),$$

$$S_{\mu\sigma}{}^{\sigma} = 0.$$

- $\Delta_\lambda{}^{\mu\nu} = 0 \Rightarrow S_{\mu\nu}{}^{\lambda} = 0$ ($\Rightarrow$ Palatini $f(R)$ gravity)
- if, additionally, $f(\mathcal{R}) = \mathcal{R}$ then GR

Vacuum and Electrovacuum

In electrovacuum both $T = T^{\mu}_{\mu} = 0$ and $\Delta_{\lambda}^{\ \mu\nu} = 0$

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T^{EM}_{\mu\nu}$$

$$\frac{1}{\sqrt{-g}}\left[-\nabla_{\lambda}\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \nabla_{\sigma}\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma(\mu}\right)\delta^{\nu)}_{\lambda}\right] = 0$$

Taking the trace again we derive just like in vacuum
$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0 \Rightarrow \mathcal{R} = c_i$ and $\nabla_{\sigma}\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}\right) = 0$

Field equations

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{4}c_i g_{\mu\nu} = \kappa' T^{EM}_{\mu\nu}, \quad \Gamma^{\lambda}_{\ \mu\nu} = \left\{{}^{\lambda}_{\ \mu\nu}\right\}, \quad \kappa' = \kappa/f'(c_i)$$

Formally equivalent to Einstein equations but $\kappa \neq \kappa'$

Perfect fluid with no vorticity (and scalar field)

$\longrightarrow$ No dependence on ${\Gamma^\lambda}_{\mu\nu}$: $\quad {\Delta_\lambda}^{\mu\nu} = 0$

$\longrightarrow$ Stress energy tensor: $T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$

Trace

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T = -\kappa(\rho - 3p)$$

Standard cases:

- Dust: $p = 0 \quad \Rightarrow \quad f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = -\kappa\rho$
- Radiation:
  $\rho = 3p \quad \Rightarrow \quad f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0 \quad \Rightarrow \quad \mathcal{R} = c_i$

Matter and torsion

In general

- $\Delta_\lambda{}^{[\mu\nu]} \neq 0$ implies $S_{\mu\nu}{}^\lambda \neq 0$ and matter introduces torsion
- $\Delta_\lambda{}^{(\mu\nu)} \neq 0$ implies that matter also introduces non-metricity

Outcome

- Gauge fields (massless particles) will not introduce torsion
- Particles with spin will generally introduce torsion
- Dirac fields are typical examples

Equivalence Principle(s)

Einstein Equivalence Principle (WEP):

- If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition.
- the outcome of any local non-gravitational test experiment is independent of the velocity of the freely falling apparatus (Local Lorentz Invariance or LLI)
- the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed (Local Position Invariance or LPI).

Metric Postulates

The metric postulates can be stated in the following way:

1. there exists a metric $g_{\mu\nu}$ (second rank non degenerate tensor).
2. $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ is the covariant derivative defined with the Levi-Civita connection of this metric and $T_{\mu\nu}$ is the stress-energy tensor of non-gravitational (matter) fields.

Main problem

Representation dependence!

Subtle point about the EP

1. What exactly is a "test particle"?
    - How small is it?
    - Can it be defined in all theories?
2. What is the relation of the EP and the variables used to describe the theory?

Main problem

EP is qualitative not quantitative: of little practical value.

Questions raised on the metric postulates

What is precisely the definition of $T_{\mu\nu}$?

- Reference to an action? Minimal coupling?
- Generalization of the special relativistic $T_{\mu\nu}$?
- A mixed definition?

What does "non-gravitational field" mean?

A field minimally coupled to gravity?
Counter example:
Scalar field in $\lambda\phi^4$ theory

$$S=\int d^4x\,\sqrt{-g}\left[\left(\frac{1}{2\kappa}-\xi\phi^2\right)R-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi-\lambda\phi^4\right]$$

One loop quantization makes ξ non-zero!

Tentative definitions

Physical Theory

A coherent logical structure, preferably expressed through a set of axioms together with all statements derivable from them, plus a set of rules for their physical interpretation, that enable one to deduce and interpret the possible results of every experiment that falls within its purview.

Representation (of a theory)

A non-unique choice of physical variables between which, in a prescribed way, one can form inter-relational expressions that assimilate the axioms of the theory and can be used in order to deduce derivable statements.

The action of scalar-tensor theory

$$S = S^{(g)} + S^{(m)}\left[e^{2\alpha(\phi)}g_{\mu\nu}, \psi^{(m)}\right]$$

where

$$S^{(g)} = \int d^4x\,\sqrt{-g}\left[\frac{A(\phi)}{16\pi G}R - \frac{B(\phi)}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right]$$

- 4 unspecified functions A, B, V, and α
- Action describes class of theories
- Obvious redundancies; fixing leads to pin-pointing either the theory or the representation!
- Action formally conformally invariant

Fixing theory or representation

- Invariance under the transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}$$

  implies that fixing any of A, B, V, and α just corresponds to a choice of Ω.
- One can conveniently redefine the scalar ϕ as well

Outcome

Two of the four function can be fixed without choosing the theory! (freedom to choose clocks and rods)

Fixing the matter fields

One could even redefine ψ as

$$\tilde{\psi} = \Omega^s \psi$$

so that

$$S^{(m)} = S^{(m)}\left[\tilde{g}_{\mu\nu}, \tilde{\psi}\right]$$

Together with the choice $A = B = 1$ the action is

$$S = \int d^4x \sqrt{-g}\left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] + S^{(m)}\left[\tilde{g}_{\mu\nu}, \tilde{\psi}\right]$$

GR + minimally coupled scalar field except $\tilde{\psi} = \tilde{\psi}(\tilde{\phi})$!!!

Jordan frame vs Einstein Frame

Jordan frame ($A=\phi$, $\alpha=0$)

$$S = S^{(g)} + S^{(m)}\left[g_{\mu\nu}, \psi^{(m)}\right]$$

$$S^{(g)} = \int d^4x\, \sqrt{-g}\left[\frac{\phi}{16\pi G}R - \frac{B(\phi)}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right]$$

Einstein frame ($A=B=1$)

$$S = S^{(g)} + S^{(m)}\left[e^{2\tilde{\alpha}(\phi)}\tilde{g}_{\mu\nu}, \psi^{(m)}\right]$$

$$S^{(g)} = \int d^4x\, \sqrt{-\tilde{g}}\left[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \tilde{V}(\phi)\right]$$

Energy Conservation

Stress-energy tensor:

Jordan frame

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{\mu\nu}}$$

Einstein frame

$$\tilde{T}_{\mu\nu} \equiv -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S^{(m)}}{\delta \tilde{g}^{\mu\nu}}$$

$$\nabla_\mu T^{\mu\nu} = 0,$$

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} = -\tilde{T}\,\frac{\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha \Omega}{\Omega}$$

Metric postulates not satisfied by $\tilde{T}_\mu \nu$ even though the two representation describe the same theory!!!

Free-fall trajectories

Considering a dust fluid in the Einstein frame with

$$\tilde{T}_{\alpha\beta} = \tilde{\rho}\, \tilde{u}_\alpha \tilde{u}_\beta$$

gives

$$\tilde{\nabla}_\alpha \left(\tilde{\rho}\, \tilde{u}^\alpha \tilde{u}^\beta \right) = \tilde{\rho}\, \frac{\tilde{g}^{\alpha\beta}\, \tilde{\nabla}_\alpha \Omega}{\Omega}$$

Projecting onto the 3-space orthogonal to $\tilde{u}^\alpha$ yields

$$\tilde{a}^\gamma = \delta^{\gamma\alpha} \frac{\partial_\alpha \Omega(\phi)}{\Omega(\phi)}$$

- No geodesic motion
- Always a force proportional to $\nabla^\mu \phi \Rightarrow$ No massive test particle in the Einstein frame!

Wrong stress-energy tensor?

Reconsider:

$$\bar{S}^{(m)} = \int d^4x \, \sqrt{-\tilde{g}} \left[-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \tilde{V}(\tilde{\phi}) \right] + \\ + S^{(m)} \left[e^{2\tilde{\alpha}(\tilde{\phi})}\tilde{g}_{\mu\nu}, \psi^{(m)} \right]$$

$$\bar{T}_{\mu\nu} \equiv -(2/\sqrt{-\tilde{g}})\delta\bar{S}^{(m)}/\delta\tilde{g}^{\mu\nu}$$

Field equations

$$\tilde{G}_{\mu\nu} = \kappa \bar{T}_{\mu\nu}$$

Bianchi identity

$$\tilde{\nabla}_\mu \tilde{G}^{\mu\nu} = 0 \Rightarrow \tilde{\nabla}_\mu \bar{T}^{\mu\nu} = 0$$

Wrong stress-energy tensor?

Not a solution!

- $\tilde{g}_{\mu\nu}$ is still not the metric whose geodesics coincide with free-fall trajectories
- $\bar{T}_{\mu\nu}$ does not reduce to the special relativistic SET when $\tilde{g}_{\mu\nu}$ is taken to be flat

$$\bar{T}_{\mu\nu} = \tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\tilde{\phi}\tilde{\nabla}_\sigma\tilde{\phi} - \tilde{g}_{\mu\nu}\tilde{V}(\tilde{\phi}) + \tilde{T}_{\mu\nu}$$

Moral

Finding quantities that satisfy the metric postulates does not mean that they will be physically meaningful

Matter or Geometry?

Example: Is ϕ a gravitational or a non-gravitational field?

- Jordan frame: Non-minimally coupled to gravity and minimally coupled to matter
  Seems gravitational!
- Einstein frame: Minimally coupled to gravity and non-minimally coupled to matter
  Seems non-gravitational!

How about vacuum?

$$\begin{aligned}\tilde{R}_{\alpha\beta} =& R_{\alpha\beta} - 2\nabla_\alpha\nabla_\beta(\ln\Omega) - g_{\alpha\beta}g^{\gamma\delta}\nabla_\gamma\nabla_\delta(\ln\Omega)\\ &+ 2(\nabla_\alpha\ln\Omega)(\nabla_\beta\ln\Omega) - 2g_{\alpha\beta}g^{\gamma\delta}(\nabla_\gamma\ln\Omega)(\nabla_\delta\ln\Omega)\end{aligned}$$

Vacuum solutions are mapped to non-vacuum solutions!

$f(R)$ gravity and Brans-Dicke theory

Introduction of an auxiliary scalar plus field redefinitions yields:

- Metric $f(R) \to \omega_0 = 0$ Brans-Dicke theory:

$$S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,[\phi R - V(\phi)] + S_M(g_{\mu\nu},\psi)$$

- Palatini $f(R) \to \omega_0 = -3/2$ Brans-Dicke theory:

$$S_{pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\phi R + \frac{3}{2\phi}\nabla_\mu\phi\,\nabla^\mu\phi - V(\phi)\right] + S_M(g_{\mu\nu},\psi)$$

Conclusions

- Problem not specific to conformal transformations
- In the $f(R)$ representations ϕ is not even there!

Eistein-Cartan(-Sciama-Kibble) theory

Description

- Theory with independent non-symmetric connection (zero non-metricity)
- Matter action depends onmetric and connection
- Two objects describing matter fields: $T_{\mu\nu}$ and $\Delta_{\mu\nu}$
- $T_{\mu\nu}$ is not divergence free

However

- $T_{\mu\nu}$ does not reduce to the SR SET at the suitable limit
- There exists a non-trivial combination of $T_{\mu\nu}$ and $\Delta_{\mu\nu}$ that does
- This combination is divergence free with respect to a third connection!

Discussion

Conclusions:

- A theory should not be identified with its representation
- Each representation can be from convenient to misleading according to the application
- Literature is biased (or even wrong in some cases)
- Definitions and common notions such as the SET, gravitational fields or vacuum are representation dependent
- Abstract statement such as the EEP are representation independent
- Precise statement such as the metric postulate are not!

Discussion

Further comments:

- Problem not confined to conformal representations
- Measurable quantities are conformally invariant, (classical) physics is not!
- Notice the analogy with coordinate independence.
- All of the above predispose us towards specific theories
- Critical obstacle for further progress

Further understanding is essential to go beyong a trail-and-error approach to gravity theories