Free float frames:

Start with a mass M which has Newtonian gravitational potential

$$\Phi = -\frac{GM}{r}.$$

Take a bunch of free float [free fall, inertial] frames out at infinity that are stationary, and drop them.

In the Newtonian approximation these free float [free fall] frames pick up a speed

$$\vec{v} = -\sqrt{\frac{2GM}{r}}\hat{r}.$$

In the free float frames, physics looks simple, and the invariant interval is simply given by

$$\mathrm{d}s_{FF}^2 = -c^2\,\mathrm{d}t_{FF}^2 + \mathrm{d}x_{FF}^2 + \mathrm{d}y_{FF}^2 + \mathrm{d}z_{FF}^2.$$

where I want to emphasise that these are locally defined free-fall coordinates.

Rigid frame:

Let's try to relate this to a rigidly defined surveyor's system of coordinates that is tied down at spatial infinity.

Call these coordinates t_{rigid}, x_{rigid}, y_{rigid}, and z_{rigid}.

Since we know the speed of the freely falling system with respect to the rigid system, and we assume velocities are small we can write an approximate Galilean transformation

$$\mathrm{d}t_{rigid} = \mathrm{d}t_{FF};$$

$$\mathrm{d}\vec{x}_{rigid} = \mathrm{d}\vec{x}_{FF} + \vec{v}\,\mathrm{d}t_{FF}.$$

Inverting

$$\mathrm{d}t_{FF} = \mathrm{d}t_{rigid};$$

$$\mathrm{d}\vec{x}_{FF} = \mathrm{d}\vec{x}_{rigid} - \vec{v}\,\mathrm{d}t_{rigid}.$$

Approximate "metric":

Substituting

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t_{rigid}^2 + ||\mathrm{d}\vec{x}_{rigid} - \vec{v}\, \mathrm{d}t_{rigid}||^2$$

Expanding

$$\mathrm{d}s^2 = -[c^2 - v^2]\, \mathrm{d}t_{rigid}^2 - 2\vec{v}\cdot \mathrm{d}\vec{x}\, \mathrm{d}t_{rigid} + ||\mathrm{d}\vec{x}_{rigid}||^2.$$

Substituting

$$\mathrm{d}s^2 = -\left[c^2 - \frac{2GM}{r}\right] \mathrm{d}t_{rigid}^2 - 2\sqrt{\frac{2GM}{r}}\, \mathrm{d}r_{rigid}\, \mathrm{d}t_{rigid} + ||\mathrm{d}\vec{x}_{rigid}||^2.$$

This is only an approximation — Newton's gravity; Galilean coordinate transformations.

The miracle du jour:

The invariant interval

$$\begin{aligned} \mathrm{d}s^2 &= -\left[c^2 - \frac{2GM}{r}\right] \mathrm{d}t_{rigid}^2 \\ &\quad -2\sqrt{\frac{2GM}{r}} \mathrm{d}r_{rigid}\, \mathrm{d}t_{rigid} \\ &\quad +\|d\vec{x}_{rigid}\|^2. \end{aligned}$$

is an exact solution of Einstein's equations of general relativity.

If you don't believe me, feed it to Maple and have it calculate the "Ricci tensor".

This is *one* representation of the space-time geometry of a black hole, in a particular coordinate system (the Painleve–Gullstrand coordinates).

There are many other coordinates systems you could use.

Schwarzschild radius:

You can see that something goes wrong at

$$\frac{2GM}{r_s} = c^2;$$

$$r_S = \frac{2GM}{c^2}.$$

Reverend John Mitchell (1783);
Peter Simon Laplace (1799).

Check dimensions!

In Einstein's gravity the coefficient of $\mathrm{d}t^2_{rigid}$ goes to zero at the Schwarzschild radius; in Newton's gravity the escape velocity

$$v_{escape} = \sqrt{\frac{2GM}{R}}.$$

reaches the speed of light once $R = r_S$.

Other versions of Schwarzschild:

Coordinate freedom in GR can lead to a lot of confusion. Consider for instance:

$$\mathrm{d}s^2 = -(1-2M/r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2\mathrm{d}\Omega^2$$

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{1+2M/r} + (1+2M/r)\,\mathrm{d}r^2 + [r+2M]^2\mathrm{d}\Omega^2$$

$$\mathrm{d}s^2 = -(1-2M/r)\,\mathrm{d}t^2 \pm 2\sqrt{2M/r}\,\mathrm{d}t\,\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + (2M/r)[\mathrm{d}r \pm \mathrm{d}t]^2$$

and

$$\mathrm{d}s^2 = \frac{16M^2}{R(t,r)}\exp(-R(t,r)/2M)\left[-\mathrm{d}t^2 + \mathrm{d}r^2\right] + R(t,r)^2\mathrm{d}^2\Omega$$

subject to

$$t^2 - r^2 = -(R-2M)\exp(R/2M)$$

These are all the *same* spacetime geometry — the Schwarzschild solution.

Kerr:

For a rotating black hole the Kerr solution was discovered in 1963 — that's 48 years after the field equations were first developed.

One version is:

$$
\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&\quad + \frac{2MR^3}{R^4 + a^2 z^2}\Big[\mathrm{d}t + \frac{R}{a^2 + R^2}(x\ \mathrm{d}x + y\ \mathrm{d}y) \\
&\quad + \frac{a^2}{a^2 + R^2}(y\ \mathrm{d}x - x\ \mathrm{d}y) + \frac{z}{R}\mathrm{d}z\Big]^2
\end{aligned}
$$

subject to $R(x, y, z)$ being implicitly determined by:

$$x^2 + y^2 + z^2 = R^2 + a^2 \left[1 - \frac{z^2}{R^2}\right].$$

Calculations using the Kerr solution are simply horrendous.