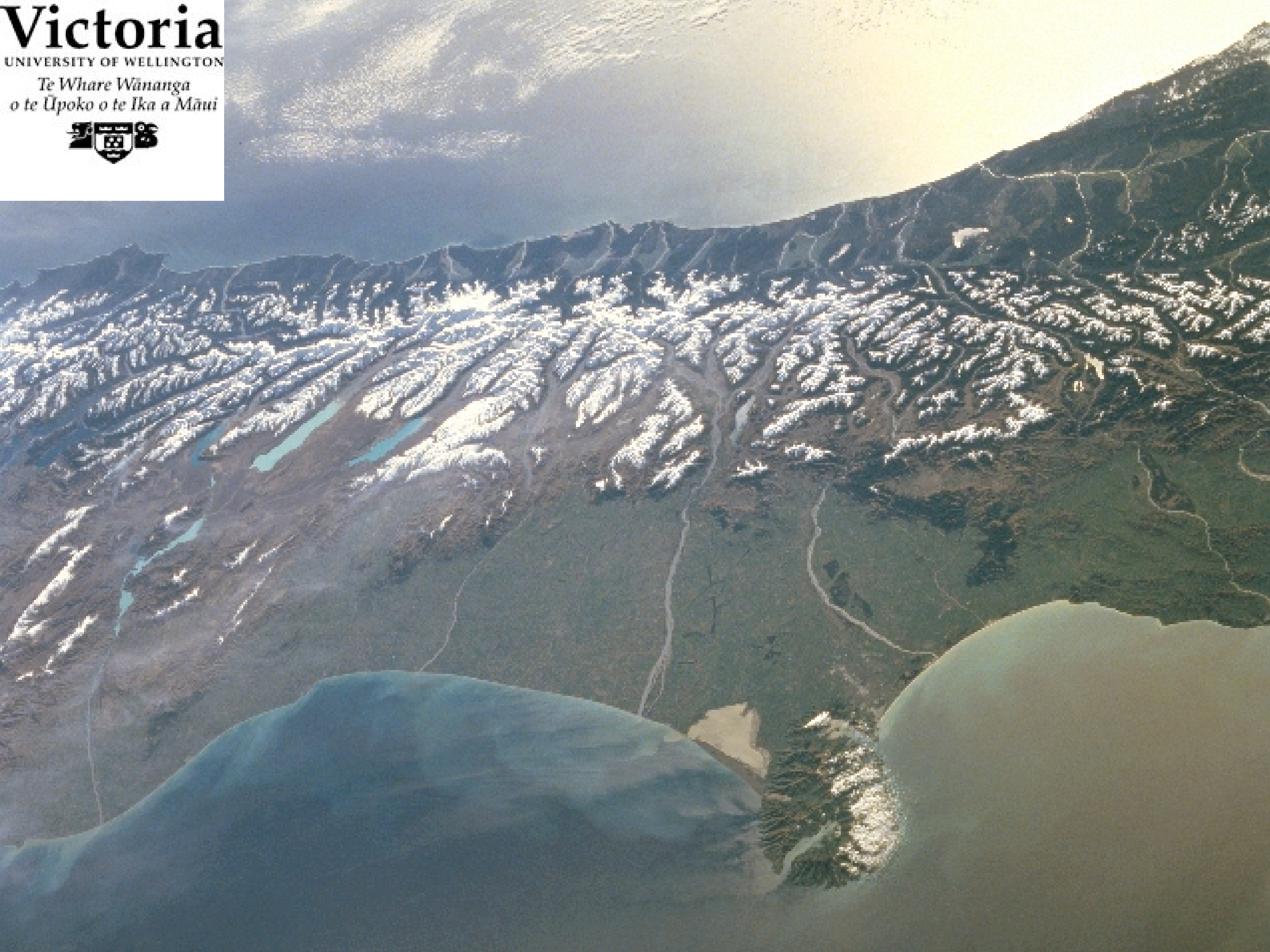
Victoria
UNIVERSITY OF WELLINGTON
Te Whare Wānanga
o te Ūpoko o te Ika a Māui



Adding vorticity:

Victoria
UNIVERSITY OF WELLINGTON
Te Whare Wānanga
o te Ūpoko o te Ika a Māui

Deriving a wave equation for sound in the presence of background vorticity is actually rather difficult.

Need Clebsch decomposition --- multiple scalar potentials.

Any vector field in three dimensions $\mathbf{v}_0 = \nabla\phi_0 + \beta_0 \nabla\gamma_0.$

Fluctuations:

$$\begin{aligned}\mathbf{v}_1 &= \nabla\phi_1 + \beta_0\nabla\gamma_1 + \beta_1\nabla\gamma_0 \\ &= \nabla(\phi_1 + \beta_0\gamma_1) - \gamma_1\nabla\beta_0 + \beta_1\nabla\gamma_0 \\ &\equiv \nabla\psi_1 + \xi_1.\end{aligned}$$

Constraint: $\xi_1 \cdot (\nabla \times \mathbf{v}_0) = 0.$

Adding vorticity:

Victoria University of Wellington
Te Whare Wānanga o te Ūpoko o te Ika a Māui

The PDEs governing linearized fluctuations (sound) are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\psi_1\right) = \frac{1}{\rho_0}\nabla\left(\rho_0(\nabla\psi_1 + \xi_1)\right),$$

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}t} = \nabla\psi_1 \times \omega_0 - (\xi_1 \cdot \nabla)\mathbf{v}_0.$$

If the vorticity is zero you recover the previous formalism.

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi\right) = 0.$$

If the wavelength and period of the sound wave are small compared to variations in the background flow, you recover Peirce's approximate equation.

Adding vorticity:

Technical details:

For any barotropic fluid:

$$S = \int \mathrm{d}t\, \mathrm{d}^3x \left\{ -\frac{1}{2}\rho \mathbf{v}^2 - \phi(\dot\rho + \nabla\cdot(\rho\mathbf{v})) + \rho\beta(\dot\gamma + (\mathbf{v}\cdot\nabla)\gamma) + u(\rho) \right\}.$$

Vary the velocity field v:

$$\mathbf{v} = \nabla\phi + \beta\nabla\gamma.$$

Algebraically eliminate the velocity field v:

$$S_{\text{new}} = \int \mathrm{d}t\, \mathrm{d}^3x \left\{ \frac{1}{2}\rho(\nabla\phi + \beta\nabla\gamma)^2 + \rho(\dot\phi + \beta\dot\gamma) + u(\rho) \right\}.$$

Adding vorticity:

Now vary the remaining variables:

$$\delta\phi: \quad \dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\delta\beta: \quad \rho(\dot{\gamma} + (\mathbf{v}\cdot\nabla)\gamma) = 0 \quad \Rightarrow \quad \dot{\gamma} + (\mathbf{v}\cdot\nabla)\gamma = 0,$$

$$\delta\gamma: \quad \partial_t(\rho\beta) + \nabla(\mathbf{v}\rho\beta) = 0 \quad \Rightarrow \quad \dot{\beta} + (\mathbf{v}\cdot\nabla)\beta = 0,$$

$$\delta\rho: \quad \frac{1}{2}v^2 + \dot{\phi} + \beta\dot{\gamma} + \mu = 0,$$

Here $\mu = \mathrm{d}u/\mathrm{d}\rho$ is the specific enthalpy.

Both β and γ are advected by the motion.

These equations are still exact.

Now need to consider (linearized) fluctuations...

Adding vorticity:

Victoria
UNIVERSITY OF WELLINGTON
Te Whare Wānanga
o te Ūpoko o te Ika a Māui

Linearize:

$$\rho = \rho_0 + \epsilon\,\rho_1,$$
$$\phi = \phi_0 + \epsilon\,\phi_1,$$
$$\beta = \beta_0 + \epsilon\,\beta_1,$$
$$\gamma = \gamma_0 + \epsilon\,\gamma_1,$$

Expand the action to quadratic order in fluctuations:

$$S_{\text{new}} = S_0 + S_1 + S_2 + \cdots.$$

$$S_2 = \int \mathrm{d}t\, \mathrm{d}^3x \left\{ \frac{1}{2}\rho_0 \mathbf{v}_1^2 + \rho_1 \mathbf{v}_0 \cdot \mathbf{v}_1 + \rho_1(\dot\phi_1 + \beta_0\dot\gamma_1 + \beta_1\dot\gamma_0) + \rho_0\beta_1\dot\gamma_1 + \frac{1}{2}\frac{c^2}{\rho_0}\rho_1^2 \right\},$$

Here $\mathbf{v}_1$ is shorthand for $\nabla\phi_1 + \beta_1\nabla\gamma_0 + \beta_0\nabla\gamma_1$.

Adding vorticity:

Victoria
UNIVERSITY OF WELLINGTON
Te Whare Wānanga
o te Ūpoko o te Ika a Māui

Now simply read off the EOM and rearrange them.

Useful definitions: $\psi_1 = \phi_1 + \beta_0\gamma_1, \quad \xi_1 = \beta_1\nabla\gamma_0 - \gamma_1\nabla\beta_0.$

Useful results: $$\rho_1 = -\frac{\rho_0}{c^2}\frac{\mathrm{d}\psi_1}{\mathrm{d}t},$$

$$\frac{\partial\rho_1}{\partial t} + \mathbf{v}_0\cdot\nabla\rho_1 + \rho_1\nabla\cdot\mathbf{v}_0 + \nabla\cdot\rho_0\mathbf{v}_1 = 0,$$

$$\frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho_0\mathbf{v}_0) = 0,$$

Implies: $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\psi_1\right) = \frac{1}{\rho_0}\nabla\left(\rho_0(\nabla\psi_1 + \xi_1)\right).$$

Adding vorticity:

If we ignore the ξ_1 then we have Pierce's equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right) \frac{1}{c^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right) \psi_1 = \frac{1}{\rho_0} \nabla(\rho_0 \nabla \psi_1).$$

By using the background continuity equation:

$$\left(\frac{\partial}{\partial t} + \nabla \cdot \mathbf{v}_0\right) \frac{\rho_0}{c^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right) \psi_1 = \nabla(\rho_0 \nabla \psi_1)$$

where each nabla now acts on everything to its right...
but this is now equivalent to:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\; g^{\mu\nu} \partial_\nu \psi_1\right) = 0, \qquad \sqrt{-g}\; g^{\mu\nu} = \frac{\rho_0}{c^2} \begin{pmatrix} 1, & \mathbf{v}_0^T \\ \mathbf{v}_0, & \mathbf{v}_0 \mathbf{v}_0^T - c^2 I \end{pmatrix}.$$

Adding vorticity:

As usual:

$$g_{\mu\nu} = \frac{\rho_0}{c} \begin{pmatrix} c^2 - v_0^2, & v_0^T \\ v_0, & -I \end{pmatrix}.$$

(overall minus sign irrelevant)

Spacetime interval:

$$ds^2 = \frac{\rho_0}{c} \left\{ c^2 dt^2 - \delta_{ij} \left(dx^i - v_0^i dt \right) \left(dx^j - v_0^j dt \right) \right\}.$$

The "scalar part" of the velocity perturbation still "sees" the same "acoustic metric", though the "vortex part" of the velocity perturbation now acts as a source:

$$\Delta_g \psi_1 = \frac{1}{\rho_0^2 \, c_0} \frac{\partial}{\partial x^i} \left(\rho \, \xi_1^i \right).$$

Victoria
UNIVERSITY OF WELLINGTON
Te Whare Wānanga
o te Ūpoko o te Ika a Māui

Adding vorticity:

To complete the job you need an EOM for ξ_1.

A brief but turgid agony leads to:

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}\mathbf{t}} = \nabla\psi_1 \times \omega_0 - (\xi_1 \cdot \nabla)\mathbf{v}_0.$$

so gradients in the "scalar part" of the velocity perturbation excite the "vortex part" of the velocity perturbation...

So even in the presence of vorticity, the "acoustic metric" is still part of the analysis --- however it is no longer the only relevant feature, with the vorticity and scalar parts of the flow now feeding each other...

Victoria
UNIVERSITY OF WELLINGTON
Te Whare Wānanga
o te Ūpoko o te Ika a Māui